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ALTERNATIVE APPROACHES TO THE EXPLANATION
OF MACROECONOMIC FLUCTUATIONS

by

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CHAPTER 1) Alternative Approaches to the Explanation of Macroeconomic Fluctuations: An Introduction

1.1. Three Alternative Approaches

The main purpose of this work is to appraise and compare alternative approaches to the study of macroeconomic fluctuations. More specifically, our purpose is to compare the advantages and disadvantages of some of the most recent approaches with those of the approach implicit in the work of pioneers of business cycle modelling such as Frisch, Tinbergen, Goodwin, Kaldor, Hicks, and Kalecki.

In doing this, we feel that the most salient distinction between these alternative approaches to the explanation of macro-economic fluctuations is the fact that, whereas the latter is macrodynamic from the very outset, the former are based on the analysis of the behaviour of optimizing agents.

In this context, "MOST RECENT APPROACHES" to the explanation of macroeconomic fluctuations can be taken to cover both equilibrium

models of the business cycle and the dynamic version of disequilibrium models of the fixed price type. Indeed, in both approaches the starting point is the analysis of maximization problems solved by economic agents and, in both approaches, this is the result of a demand for better micro-foundations on which to build a sound macro-economics¹. For example, in Lucas' opinion, i.e., in the opinion of the most representative exponent of the equilibrium approach, the opinion according to which macroeconomics is in need of a microeconomic foundation is nowadays a commonplace. This means, for example, that the observed movements in quantities (e.g, investment, consumption, or employment) should be explained as optimizing agents' responses to observed movements in prices².

1. This fact is clearly acknowledged in the existing literature on the topic. For example, as stressed by Solow, in fixed price models economic agents are assumed to optimize under the circumstances that they perceive to be prevailing. The difference with models in the equilibrium approach is that

"... among the constraints that they perceive are market constraints, an inability to sell (or to buy) what they would like to sell or (buy) at going prices. ... The difference between the equilibrium view and the disequilibrium view is not that in one theory agents are assumed to optimize and in the other they are not. The difference is in the constraints they are assumed to take into account"

(Solow, 1979, p.345; original emphasis).

This is shown, for example, by Fitoussi (1983; pp.8-19) who defines four concepts of temporary equilibrium, among which the Walrasian equilibrium and the Non-Walrasian equilibrium with fixed prices: the different concepts are obtained as solutions of the same maximization problem which, however, in each case, is solved under different constraints. See also Velupillai, 1986, pp.269-270.

2. See Lucas, 1977, p.216 and 222.

On the other side, for example in Böhm's opinion, i.e., in the opinion of the author who made one of the earliest attempts to dinamize a fixed price model, current macroeconomic models must be criticized because of their use of a large list of ad hoc assumptions about the behaviour of economic agents which lack a rigorous microeconomic foundation¹.

An important consequence of this kind of attitude is that in models within these two approaches the aggregate outcome is "mimicked" by the consideration of the interaction between representative households and firms².

Bearing this in mind, it is natural to oppose these most recent approaches with what we could refer to as the (disequilibrium) Macro-dynamic approach, namely, the approach which, starting with Frisch, covers the contributions to business cycle theory of GOODWIN, KALDOR, HICKS, and KALECKI. Although these are all very different contributions, there is an important unifying element which is given by the fact that the analysis is carried out at the macro level as distinct from the micro level. For example, this is what allows us to link Kalecki's and Frisch's linear contributions to the basic nonlinear contributions by Goodwin, Kaldor, and

1. See Böhm, 1978. p.179.

2. See Lucas, 1980, p.711.

Hicks¹.

To make any kind of comparison between the most recent approaches and the (disequilibrium) macrodynamic approach, however, does not seem to be an easy task because, although they are all labeled "business cycle theories", in fact the subject under study appears to be different.

In the (disequilibrium) macrodynamic approach, for example, one of the main problems is to explain (endogenously) the persistence of the cycle and, as a consequence, attention is principally concentrated on mechanisms which can explain endogenously the turning points, especially, the lower turning point of the cycle². For the equilibrium approach, on the contrary, to find a

1. Although Frisch's name is usually associated with exogenous explanations of the business cycle where the persistence of the cycle is accounted for by erratic shocks as in the equilibrium approach, it is not too difficult to understand in which sense he can be seen as one of the pioneers of the opposite approach. After all, as stressed by Samuelson (1943, p.284), it was with his 1933 paper on propagation and impulse problems that the upheaval from statical to dynamical point of view started. Moreover, it is in his paper that the expression "macro-dynamic analysis", as opposed to "micro-dynamic analysis", is introduced. In Frisch's opinion, whereas the micro-dynamic analysis is devoted to the explanation in some detail of the behaviour of a certain market, taking as given all general parameters, the macro-dynamic analysis must account for the fluctuations of the whole economy taken in its entirety. In order for this to be possible, in macro-dynamics it is necessary to deliberately disregard a considerable number of the details of the picture.

2. See, for example, Kalecki, 1954, p.127; Goodwin, 1955, pp.218-219; Hicks, 1950, pp.98-105; Kaldor, 1940. The importance, and difficulty, of accounting for the lower turning point of the cycle is well explained in Matthews' 1959 book on trade cycle theory where a whole Chapter (Ch.X, pp.160-178) is devoted to the analysis of this problem.

solution to this problem is not very important given that the dynamics of the economy is mainly accounted for by random shocks¹. Moreover, even the expression "regular" cycle takes on a different meaning. Indeed, in these models, given the role played by random shocks, fluctuations are highly irregular and the only possibility that still remains of interpreting them as constituting a "regular" cycle is that of detecting regularities in the "comovements" among economic variables. For example, this is stated clearly by Lucas in his review of the qualitative features of the economic time series which he calls "the business cycle"². In his opinion, economic fluctuations do not exhibit uniformity of either period or amplitude and the only regularities which are observed are in the comovements among different aggregative time series. It is with respect to the qualitative behaviour of these co-movements

1. The difference between the two definitions of the object under study in business cycle theory is stressed also by Fitoussi and Velupillai. In their 1984b paper (p.6), for example, we read that in Lucas' model (1975),

"...The business cycle itself is defined to be fluctuations in output about a trend and associated co-movements in normal rates of interest. The old-fashioned question of stagnation and the problem of the 'lower turning point' does not arise in such a model because of price and wage flexibility and the existence of equilibrium".

In particular,

"... The absence of emphasis on flow variables will be disturbing from those trained in the traditional multiplier-accelerator version of cycle theory".

2. See Lucas, 1977, pp.9-10.

among series that business cycles are all alike¹.

In this sense, business cycle theory can be seen as having come to a standstill: on the one hand, there are "old-fashioned" models which give a stylized representation of periodic macroeconomic fluctuations characterizing the dynamics of capitalist economies. On the other hand, there are more recent models which, in describing irregular fluctuations due mainly to exogenous shocks or in formulating the dynamics of the economy in terms of switchings from one regime to another, do not pay any attention to problems which were previously of the greatest importance².

1. As we read in Lucas' paper, the principal among these regularities are the following:

"...(i) output movements across broadly defined sectors move together. ... (ii) Production of producer and consumer durables exhibits much greater amplitude than does the production of nondurables. (iii) Production and prices of agricultural good and natural resources have lower than average conformity. (iv) Business profits show high conformity and much greater amplitude than other series. (v) Prices generally are procyclical. (vi) Short-term interest rates are procyclical; long-term rates slightly so. (vii) Monetary aggregates and velocity measures are procyclical"

(Lucas, 1977, p.10).

2. We have already mentioned the differential importance which is given by the different approaches to the problem of explaining endogenously the persistence of the cycle. Another example is given by the differential importance which, in the three approaches, is given to the problem of the interaction between cycles and growth trend. In the opinion of authors belonging to the (disequilibrium) macrodynamic approach, this is a basic problem for business cycle modelling. The importance which was given to this problem can be understood taking account of the fact that the cycle-trend problem, together with the problem of endogenously explaining the

(Footnote continues on next page)

A first step toward overcoming this unpleasant situation can be taken by discussing the problem of what the object of the theory of economic fluctuations is.

One possible answer to this question is, following Haberler, to say that, although each cycle has its special features different from those of any other cycle, we can evolve a very general theory of the most important aspects of the cycle. From this point of view, the object of the theory of economic fluctuations becomes that of explaining and describing the "typical business cycle" which has its roots in endogenous mechanisms governing the accumulation process of the capitalist economy and which can be discerned by abstracting from the diversities specific to each single cycle¹. The alternative answer is to say that the object of the theory of economic fluctuations must be precisely the explanation of the observed irregularities.

There have already been contributions to business cycle theory which confront this basic question.

(Footnote continued from previous page)

persistence of the cycles, is what explains the dissatisfaction with linear models of the business cycle and the consequent passage to a nonlinear formalization. On the contrary, the problem of the interaction between cycle and trend is not even mentioned by authors belonging to the most recent approaches.

1. See Haberler, 1964, pp.274-277.

According to Candela and Gardini¹, for example, these two alternative views on the object of the theory of economic fluctuations have lead to the situation we have described above in which there are theories of the business cycle with very different contents. In their opinion, the "new classical" macroeconomics has made the mistake of analysing irregular fluctuations which are due to random shocks and which, as a consequence, do not require an endogenous explanation of their turning points. On the contrary, the correct solution to the problem is to construct a model which can both explain endogenously the turning points of the cycle and describe irregular fluctuations, namely, a model which can produce chaotic dynamics².

This problem is not tackled any longer by the two authors who, instead, prefer to concentrate their attention on the problem which is at the root of all the others, namely, the problem of trying to understand whether the hypothesis of chaotic dynamics is supported or not by the empirical evidence. As stressed by them, empirical investigations show fluctuations in economic activity which are recurrent but not periodic³; however, the observed "disorder"

1. See Candela-Gardini, 1984. For a short version of this paper in English, see Candela- Gardini, 1986, where the main conclusions of the 1984 article are summarized.

2. For applications of the theory of chaos to business cycle theory, see the references given in Candela-Gardini, 1984, p.567n.

3. For example, this is shown by Candela and Gardini themselves which, for their empirical investigation, utilize data on Italian GDP for the period running from 1960 to 1982.

could be the result of the operation of random shocks. As a consequence, it is necessary to have a model which allows one to distinguish between the two alternatives. The model considered by Candela and Gardini generates a first order nonlinear difference equation with a stochastic component which describes the dynamics of national income. The estimation of the parameters of this equation demonstrates the validity of its nonlinear formulation although the estimated values are far from the values which would lead to a chaotic dynamics. Thus, the conclusion that Candela and Gardini draw is that, although in the period considered in their study it has been possible to discern aperiodic oscillations which have characterized the dynamics of the Italian economy, their irregularities are explained by the stochastic component.

The approach to the problem chosen by Candela and Gardini is surely interesting and may clarify the problem of the question of what is the object of the theory of economic fluctuations. However, my feeling is that following an analogous strategy we could not go very far in the analysis of the advantages and disadvantages of the different approaches to business cycle modelling. The reason for this is that what significantly distinguishes the various models belonging to the different approaches is not the fact that the dynamics of the system is explained by stochastic or deterministic components or that the resulting fluctuations are regular or irregular. On the contrary, these are all results that, to some extent, can characterize all different approaches. The problem, in our case, is therefore that of finding an alternative criterion

which allows one both to compare the different approaches and to focus on the crucial dissimilarities.

As we have already stressed, given that even the definitions of the business cycle are different in the different approaches, this is not an easy task to accomplish because it seems impossible to find a general aspect, common to all theories, that allows one to perform any kind of comparison. However, in our opinion, it is possible to make a first step in this direction by considering the only aspect of the problem which seems to be relevant with regard to economic modelling in general and therefore also with regard to the three different approaches to business cycle modelling we have chosen to compare, namely, the role which, in the various models, is played by the specification of "ad hoc" assumptions¹.

There are many reasons why I think this is a task worth being accomplished.

Firstly, very recent contributions to business cycle theory, and, more in general to economic dynamics, make a reflection on this problem necessary. Indeed, in many recent contributions, both within the equilibrium approach and within the fixed price

1. An hint in this direction is given by Solow in his paper on alternative approaches to macroeconomics when he says that, to discuss and compare the main schools of thought in contemporary macroeconomics,

"... it is very important to keep straight the preconceptions that underlie each one. Otherwise it is too easy to accept quite powerful conclusions without a clear grasp of the assumptions to which they are logically bound".

(Solow, 1979, p.340; my emphasis)

approach, what we have called the (disequilibrium) macrodynamic approach is summarily dismissed because of its reliance on "ad hoc" assumptions. As is clear from these contributions, the expression "ad hoc assumptions" is taken to mean assumptions which lack microeconomic foundations in the traditional sense. Thus, taking account of the very general meaning of the Latin expression "ad hoc"¹, it becomes interesting to analyse the same problem, i.e., the problem of the role played by "ad hoc" assumptions, also with regard to the most recent approaches to business cycle modelling. To do this may be an important first step for a coherent comparison of the alternative approaches. In particular, our impression is that an analysis of this kind can give us insights into the problem that prove to be more problematic for the most recent approaches than for the macrodynamic approach.

Secondly, an analysis of the role played by "ad hoc" assumptions also makes it possible to understand developments within each single approach, especially within the equilibrium approach. In particular, the two recent contributions to this approach made by Grandmont (1985) and by real business cycle theorists, for example by Long and Plosser (1983), make very actual the problem of the role played by "ad hoc" assumptions in equilibrium models. Given that the equilibrium approach is undoubtedly the most influential

1. Namely, "ad hoc" assumption in the sense of an assumption designed for a particular problem.

of the two most recent approaches to economic dynamics, it deserves particular attention.

Finally, to focus on the role played by "ad hoc" assumptions in the three approaches may help us in finding a terminology which stresses that the three approaches are alternative rather than complementary. Until now we have referred to the three approaches as the equilibrium approach, the (disequilibrium) macrodynamic approach and the dynamic version of the disequilibrium approach of the fixed price type. However, this terminology, though being the standard one, is, at least in part, misleading when the purpose is to compare the three alternative approaches to the explanation of such an important dynamic process as the business cycle.

For example, no problems arise in defining as the "equilibrium approach" the approach implicit in the work of Lucas & co. However, then, it would be natural to define the disequilibrium approach, as truly opposite in both spirit and form to the equilibrium approach, the approach implicit in the work of authors such as Goodwin and Kaldor¹. The problem, however, is that this choice would introduce much confusion because, according to the accepted terminology, the expression "disequilibrium macroeconomics" refers to models of the fixed price type, i.e., to models which, following the impulse of Clower (1965) and Leijonhufvud (1968), are concerned with the problem of "equilibrium with rationing". On the other

1. The reason for this is that in the work of both Goodwin and Kaldor the assumption of disequilibrium is the crucial one which, by itself, accounts for the endogenous dynamics of the economy.

hand, also the choice we have made of referring to Goodwin & co.'s approach as the macrodynamic approach is not fully satisfactory. Indeed, although this definition both stresses the truly macrodynamic character of this approach and opposes it to the disequilibrium approach of the fixed price type, it does not do full justice to the differences between this approach and the equilibrium approach. The latter, in fact, undoubtedly takes a dynamic point of view from the very outset and what mainly accounts for the dynamics are shocks, for example monetary shocks, which affect the whole economy¹.

A possible way to give a solution to these problems of terminology is, first, to clarify the object to which we want to devote our attention and, second, taking account of such an object, to introduce a new, suitable terminology - namely, to introduce an "ad hoc" terminology (!). Although this can cause even more confusion in a topic which already abounds in definitions, I think that it is both possible and useful to follow such a strategy in our case because, given the purpose of our research as we have stated it above, there is a definition of the (disequilibrium) macrodynamic approach that seems to propose itself. As we have already noted, any assumption that is not derived explicitly from an individual optimization process is condemned to be considered "ad

1. On this point, see Fitoussi-Velupillai, 1984a, pp.1-6.

hoc" by both equilibrium and disequilibrium theorists¹. Thus, the most useful definition of the (disequilibrium) macrodynamic approach to business cycle modelling is a definition which stresses the fact that, in this approach, all behavioural relations which are taken into consideration are aggregate relations - for example, total consumption as a function of national income or total investment as a function of the increase in total income - which have no immediate relation with any individual maximization process. If necessary, it is possible to defend these aggregative relationships by a "microeconomic story" and in fact, as stressed by Solow², this has often been done. However, in the (disequilibrium) macrodynamic approach this sort of "micro-rationalization" of macroeconomic relationships is made very informally and always with a view to the aggregate outcome which is at centre of the analysis and which has then to be used, together with national accounting identities, to generate the equation governing the dynamics of the economy in its entirety. In this sense, they are equations explaining the macro-behaviour so that what we have until now referred to as the (disequilibrium) macrodynamic approach may well be defined as the "macro-behavioural" approach. This definition eliminates any possibility of overlapping of the three approaches we have chosen to compare

1. From now on, "disequilibrium", without any other specification, refers to the disequilibrium approach of the fixed price type.

2. See Solow, 1986, pp.196-198.

and therefore seems to be the most useful given the object of our study.

Armed with such a definition, our purpose is now to describe how the analysis of the role played by "ad hoc" assumptions in the alternative approaches will be dealt with in each of the following chapters.

1.2. Plan of the Thesis

As we have already stressed, importance is given to the problem of the role played by "ad hoc" assumptions in business cycle models by very recent contributions. In particular, given the important contributions to the problem by Grandmont and "real business cycle" theorists, for example Long and Plosser, it seems important to analyse this problem, first of all, with regard to the equilibrium approach. For this reason, the equilibrium approach is analysed in Chapter 2.

In both Grandmont's and Long-Plosser's contributions, the purpose is to show that an equilibrium model can give rise to persistent fluctuations without having to rely on rather special

("ad hoc") assumptions¹. The model utilized by Grandmont to show this is an overlapping generations model where markets clear at every date and traders have perfect foresight. The model utilized by Long and Plosser is, instead, an equilibrium "real" model, i.e., an equilibrium model where monetary disturbances are deliberately ignored and where it is assumed that economic agents have complete information.

In order to better understand the role of these recent contributions to the equilibrium approach, first, in Chapter 2.1., some very simplified versions of "standard" equilibrium models by Barro and Lucas are presented. Our purpose in doing this is to underline the crucial ("ad hoc") role played in these models by the assumption of imperfect (current) information and by exogenous disturbances.

Then, in Chapter 2.2., Grandmont's overlapping generations model, where the resulting fluctuations are endogenous, is considered. Our purpose in doing this is to stress that also in Grandmont's model an important role is played by "ad hoc" assumptions so that, firstly, his criticism of the Lucasian model

1. More precisely, Grandmont's purpose is to show that an equilibrium model can be characterized by cyclical fluctuations that display the sort of correlations that the New Macroeconomic models have sought to incorporate "...without having to make the 'ad hoc' assumption that cycles are due to exogenous shocks" (Grandmont, 1985, p.996). Long's and Plosser's purpose, instead, is, in short, to show that even in a model with no monetary disturbances, "...certain very ordinary economic principles lead maximizing individuals to choose consumption-production plans that display many of the characteristics commonly associated with business cycles" (Long-Plosser, 1983, p.67).

does not seem to be as well founded as the author seems to believe, and, secondly, in particular because of the role played in the model by the specification of the expectation function, it gives support to the idea that equilibrium models do not really provide a convincing explanation of economic fluctuations.

To arrive at this conclusion, in Chapter 2.3. also Long-Plosser's model, where the basic role in explaining fluctuations in economic activity is played by random shocks to production transformations, is considered.

To round off the analysis of the most recent approaches to the analysis of macroeconomic fluctuations, in Chapter 3 the problem of the role played by "ad hoc" assumptions is analysed with regard to models within the disequilibrium approach. To this end, the recent contributions by Blad (1981) and by Blad and Zeeman (1982) are considered. The purpose in doing this is to show that in these models it is necessary to rely on various ("ad hoc") assumptions in order to apply the analysis to different cases of regime switching. As we will see, an interesting implicit fact about these models is that they utilize analytical tools which call to mind tools utilized in the macro-behavioural approach; indeed, the formalization is in terms of (generalized) "relaxation oscillations".

To understand this, in Chapter 4, the macro-behavioural approach is introduced with the purpose of making possible a comparison of it with the most recent approaches with regard to the problem of the role played by "ad hoc" assumptions. To this end, firstly, in Chapter 4.1, the concept of relaxation oscillations is introduced

making reference to equations of both van der Pol's and Lord Rayleigh's type. Among other things, this is useful in order to understand the importance that the existence of limit cycles can have for the representation of persistent and endogenous fluctuations of the system.

Le Corbeiller's article (1933), published in the first volume of *Econometrica*, is usually considered as the first presentation of the new nonlinear mathematical tool to economists. For example, in Goodwin's article¹ explicit reference in this sense is made to Le Corbeiller's work. However, to give a complete view of the way in which nonlinear differential equations of van der Pol's type have been introduced in business cycle modelling, we consider, in Chapter 4.2., an earlier - often neglected - contribution to the problem given by the Dutch economist L. Hamburger². Our purpose in doing this is to understand the reasons adduced by him for the introduction of relaxation oscillations in business cycle modelling. If nothing else, this is a necessary step given Hamburger's claim to "paternity" of the introduction of relaxation oscillations in business cycle theory³.

Finally, in Chapter 5, models within the macro-behavioural approach are analysed with the purpose of taking the matter of the role played by "ad hoc" assumptions up again. As was the case with

1. See Goodwin, 1951, p.228 and 228f.

2. For example, see Hamburger, 1931.

3. See Hamburger, L., 1934.

regard to the equilibrium approach, also with regard to the macro-behavioural approach importance is given to this problem by very recent contributions, in this case by Benassy¹.

In these contributions, Benassy's purpose is to show that in a nonlinear model of the business cycle, it is possible to obtain the representation of endogenous, persistent fluctuations, without having to make assumptions which in his opinion are "ad-hoc", for example, without having to assume a sigmoid shape for the investment function. In fact, his purpose is to show that such a dynamic behaviour can be obtained assuming "traditional" shapes for the various functions.

To understand the relevance of Benassy's criticism of the macro-behavioural approach, it is necessary to understand in which sense the specification of the nonlinearities in a nonlinear model of the business cycle can be "ad hoc".

In the attempt to answer this question, after having justified in Chapter 5.1. the choice of concentrating our attention explicitly only on Goodwin's and Kaldor's models, the problem of the conditions for the existence of a limit cycle in Goodwin's 1951 model is analysed in Chapter 5.2.. To this end, the Liénard criterion is utilized to obtain sufficient conditions for the existence of limit cycles².

1. See Benassy, 1984, 1986.

2. See, for example, Minorsky, 1962, Ch.4, pp.101-117. A useful introduction to the use of this criterion is given also in Jordan-Smith, 1977, pp.17-25 and 100-104.

To understand better the use of Liénard's criterion, in Chapter 5.2., it is shown that the same method can be applied to other models within the macro-behavioural approach, for example, to Kaldor's 1940 model. As we will see, an analysis of this kind gives us insights into the problem that prove to be very useful in order to evaluate the relevance of Benassy's criticism of the macro-behavioural approach. Indeed, after having understood the functioning of this method, one is immediately suspicious that the existence of a limit cycle in Benassy's model relies in the same way on an "ad hoc" assumption.

In Chapter 6, it is shown that this is in fact the case.

In Benassy's model, firstly, the equilibrium value of output (income) is determined in the short run as a function of expected demand and nominal wage. Secondly, a dynamic system describing the evolution in time of the nominal wage - through a Phillips Curve-type relation - and of the expected demand - through an adjustment mechanism - is considered. Below, the short run equilibrium model and the equations governing the dynamics of wage and expected demand are analysed in Chapter 6.1. and 6.2. respectively.

Although Benassy's purpose is to show that limit cycles can be generated in his model by using the "traditional shapes" for the various functions, it is intuitively easy to understand that the possibility of having limit cycle-type behaviour in his model arises because of the consideration of an ("ad hoc") nonlinearity in the same sense in which the formulation of the nonlinear investment functions in Goodwin's and Kaldor's models is "ad hoc".

To show this, in Chapter 6.3, first, Benassy's dynamical system is reduced to an equivalent nonlinear second order equation of the van der Pol type, and, second, Liénard's criterion is applied to this equation.

A final remark seems necessary.

Given the truly alternative character of the three approaches we have chosen to compare, that among other things ensures that each of them has very different political implications, an analysis of their advantages and disadvantages cannot be seen as having the purpose of helping in deciding whether to be in favour of adopting one approach rather than another. On the contrary, this is a decision which is always taken right at the beginning of a study. Thus, the analysis we are going to develop in the following chapters is, paraphrasing the title of Solow's article, a "partial" one as opposed to an "impartial" one in that our purpose is to try to understand whether, from the point of view of the macro-behavioural approach, the most recent approaches can nevertheless give us new insights into the problem of "understanding business cycles".

Some conclusions in this sense are drawn in Chapter 7.

The illuminating book by Gabisch and Lorenz (1987) was published after my research had already reached an advanced stage. This book

also deals with many of the arguments covered here. Therefore, at those points where there is overlap between our findings and those of Gabisch and Lorenz, reference will be made in a footnote.

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FIRST PART: THE MOST RECENT APPROACHES



CHAPTER 2: The Equilibrium Approach

2.1. Equilibrium Models of the Business Cycle with Imperfect Information and Random Disturbances

To understand the role of recent contributions to the equilibrium approach, it is useful to review some features of "standard" equilibrium models that may be thought to be unsatisfactory such as, for example, the reliance of this kind of model firstly on the assumption of imperfect information and secondly on random disturbances.

For the sake of such a review, we consider very simplified versions of "standard" equilibrium models which in leaving out more sophisticated aspects, allow us to focus on the two above mentioned features. Thus, by analysing simpler versions of the model we can easily grasp the crucial role played in the "standard" formulation of the equilibrium business cycle model by the assumption of

imperfect (current) information, i.e., by the specification of the information set, and by exogenous, random disturbances¹.

2.1.1. The Role Played by the Specification of the Information Set

In the standard version of equilibrium business cycle models, an economy in which goods and services are traded in a large number of localized markets - "islands"² - is considered. In such an economy, the "imperfect information story" can be very easily introduced by assuming that individuals, for some "unexplained" reason, can receive current information on current values of the variables only locally but not concerning the whole economy. For example, the typical assumption is that according to which individuals in market z receive current information on the local price $p_t(z)$ but only lagged information on prices in all other markets and on the economy-wide average price.

One of the formulations of the supply and demand functions often used in equilibrium models assumes that both demand for and supply of output depend on perceived excess of observed prices over

1. See, for example, Barro, 1976, 1980, 1981; Lucas, 1973, 1975, 1976.

2. To distinguish them, all variables are indexed by a parameter z ; e.g., $p_t(z)$ refers to the price at time t in market z .

anticipated normal values and on the unperceived part of current money growth.

For example, in a simplified version of Barro's model¹, these two functions, in log terms, take the following form:

$$y_t^s(z) = a_s[p_t(z) - E_z p_{t+1}] - b_s(m_t - E_z m_t) + \varepsilon_t^s(z) \quad (1),$$

$$y_t^d(z) = -a_d[p_t(z) - E_z p_{t+1}] + b_d(m_t - E_z m_t) + \varepsilon_t^d(z) \quad (2),$$

where $a_s, a_d, b_s, b_d > 0$; $p_t(z)$ is the price in market z at time t ; $E_z p_{t+1}$ is the expectation formed at time t in market z of the following period's general price level; m_t is the rate of growth of money at time t ; $E_z m_t$ is the part of money growth perceived by individual in market z ; and the $\varepsilon_t(z)$'s are local disturbances terms.

In equations (1) and (2), the term $[p_t(z) - E_z p_{t+1}]$ reflects a positive (negative) speculative response of supply (demand) to perceived excess of the price which is observed in market z over anticipated (future) general price² whereas the term $(m_t - E_z m_t)$ accounts for the effect of the unperceived part of current money

1. See Barro, 1980, 1981. To obtain a simplified version of the model we have assumed that the nominal interest rate is equal to zero.

2. In the case in which the nominal interest rate is not equal to zero ($R_t \neq 0$), it is necessary to replace the term $[p_t(z) - E_z p_{t+1}]$, which represent a kind of "anticipated one-period real rate of return" from the perspective of market z , with $[p_t(z) - E_z p_{t+1} + R_t] = r_t(z)$. See Barro, 1981, pp.42-45.

growth on supply and demand. As will appear evident, this term in the two equations is what links nominal disturbances to real variables.

Accordingly, assuming that goods cannot travel across markets within a period and that local prices are completely flexible, we obtain the following market clearing condition for market z :

$$y_t^s(z) = y_t^d(z),$$

i.e.:

$$\begin{aligned} a_s[p_t(z) - E_z p_{t+1}] - b_s(m_t - E_z m_t) + \epsilon_t^s(z) &= \\ = -a_d[p_t(z) - E_z p_{t+1}] + b_d(m_t - E_z m_t) + \epsilon_t^d(z) \end{aligned} \quad (3),$$

from which:

$$p_t(z) - E_z p_{t+1} = \left(\frac{1}{a}\right)[b(m_t - E_z m_t) + \epsilon_t(z)] \quad (4),$$

where $a = a_d + a_s$, $b = b_d + b_s$, $\epsilon_t(z) = \epsilon_t^d(z) - \epsilon_t^s(z) \sim N(0, V)$.

Finally, inserting (4) in (1) or in (2), we obtain the following equation for output:

$$\begin{aligned} y_t(z) &= \left(\frac{1}{a}\right)(a_s b_d - a_d b_s)(m_t - E_z m_t) + \\ &\quad + \left(\frac{a_s}{a}\right) \epsilon_t^d(z) + \left(\frac{a_d}{a}\right) \epsilon_t^s(z) \end{aligned} \quad (5).$$

The expressions we have obtained in (4) and (5) are not final (equilibrium) solutions for $y_t(z)$ and $[p_t(z) - E_z p_{t+1}]$ because the right-hand side of both equations contains the endogenous expectation $E_z m_t^1$. However, on the basis of these expressions it is already possible to identify some properties of this type of model. Two properties in particular of the expressions for $y_t(z)$ and $[p_t(z) - E_z p_{t+1}]$ we have obtained in (4) and (5) seem to be worth mentioning.

Firstly, as underlined by Barro himself², the monetary influence on real variables is expressed by the term $(m_t - E_z m_t)$, i.e., by the unperceived part of money growth. To notice this allows one to understand the crucial role played by the assumption of imperfect (current) information. Indeed, $E_z m_t$ is nothing but the expected money growth conditional on the information set $I_t(z)$ available in market z at time t :

1. As we will see while analysing, in the next Section, Lucas' model, the procedure usually employed in equilibrium models to obtain the solutions in $p_t(z)$ and $y_t(z)$ as functions of exogenous variables consists in applying the analogous of the method of underdetermined coefficients to a trial solution in the case of differential or difference equations. In the present case, this procedure would consist, first, in writing out the form of the solution for $p_t(z)$ in terms of unknown coefficients on the relevant independent variables. Second, in determining the values of the unknown coefficients by using the market clearing condition (3). For applications of this procedure, see, for example, Barro, 1976, 2.2. "Market-Clearing Determination of Prices and Outputs", pp.5-12; Lucas, 1973, pp.330-332, 1975.

2. See Barro, 1981, p.46.

$$E_z m_t = E[m_t | I_t(z)],$$

so that to assume that $m_t \neq E[m_t | I_t(z)]$ means to assume that the information set does not contain m_t , i.e., that it contains only the actual price in market z as current information. Clearly, a slight different specification of the information set available to individuals in any market z can lead to opposite conclusions and this seems to be an important shortcoming of this type of model¹.

Secondly, there is the important problem of accounting for the persistence of business fluctuations. Indeed, if full information is received with a one-period lag, this model cannot account for effects of money shocks on output that last more than one period. This criticism has led not only to Lucas-Sargent's counterargument² but even to further extensions of the basic model to account for persistence. Of these extensions, the classical example is given by Lucas' 1975 model where serially correlated output movements are the consequence of the following

1. See Fitoussi, 1983, pp.20-21. This shortcoming is also noticed by Barro himself who writes:

"...The solution is sensitive to the specification of individual information sets. If individuals have full current information, so that $m_t = E m_t$ applies, ... output and the anticipated real rate of return are equal to their natural values. In this case all monetary movements are neutral"

(Barro, 1981, p.49; my emphasis).

2. See Lucas-Sargent, 1979, p.313. Their counterargument is based on Frisch's distinction between sources of impulses and propagation mechanism.

two additional assumptions: 1) a lag of more than one period in information acquisition; and, 2) the presence of a stock variable which respond positively to money shocks¹. However, this, in particular the fact that persistent fluctuations can be easily

1. The role of the consideration of a stock variable, which in Lucas' model is the productive capacity, in accounting for the persistence of the effects of money shocks on output is well explained in Barro's book (1981, p.48) where we read:

"... Suppose that capacity responds positively to a money shock during the period when the shock is not fully understood. As information on nominal aggregates becomes available in future periods, individuals are likely to regret some of their earlier decisions on employment, production, investment and so on. However, these decisions are bygone that typically alter the initial conditions for future periods by changing the value of productive capacity. Since the existence of this heightened capacity is likely to imply at least a temporary outward shift in labor demand and commodity supply, the monetary shock will tend to have persisting effects on output and employment".

The problem, in Barro's opinion, is that productive capacity in real economies does not exhibit the cyclical variability which is required to represent an empirically important channel for persistence. However, the strength of Lucas' argument is given by the fact that his

"... modeling of capacity can be generalized beyond the explanation of this particular variable" (ib.). More specifically, it "... applies to any variable that responds initially to monetary shocks and then has a durable aspect that implies a change in future initial conditions"

(ib.; my emphasis)

As Barro himself stresses (ib., p.49), however, this kind of extensions of the basic equilibrium model, while accounting for the persistence of the cycle, reduce the quantitative role of other factors, e.g., of intertemporal substitution mechanism, as the basis for fluctuations in output and employment. The analogy between this kind of mechanism and the accelerator effects considered by authors in the macrodynamic tradition can immediately be seen.

accounted for by the consideration of a variable lag in the acquisition of information, is nothing but the confirmation of the crucial role that is played by the ("ad hoc") specification of the information set.

In this perspective, therefore, Grandmont's and Long-Plosser's papers, where it is assumed that individuals have full current information, can be better understood if they are seen as attempts to reduce the reliance of equilibrium models on the specification of the information set.

2.1.2. The Role Played by Exogenous Shocks in a Lucasian Model

An analysis of Lucas' model of the output-inflation trade-off is very useful in order to understand another aspect of the problem.

In Lucas's model¹, the (log of) supply in any market z at time t is supposed to be equal to the sum of a permanent component, y_t^P , which, for the sake of simplicity, is assumed to be equal in all markets, and of a cyclical component, $y_t^C(z)$, which is assumed to vary from market to market and to react to changes in the relative prices, as these changes are perceived in any market z on the basis of the available information set.

Thus, we can write:

1. See Lucas, 1973, 1976. See also Sargent, 1979, Chapter XIII, "The Phillips Curve", pp.324-332; Taylor, 1985, pp.396-402.

$$y_t(z) = y_t^P + y_t^C(z) \quad (1),$$

where:

$$y_t^C(z) = b\{p_t(z) - E[p_t|I_t(z)]\} + \lambda y_{t-1}^C(z) \quad (2),$$

with $0 < \lambda < 1$, $b > 0$.

In equation (2), as in Barro's model, the expectations operator appears because the information set $I_t(z)$ available to suppliers in market z at time t is assumed to contain only the actual price in that market as current information¹. Thus, we have the following partition of the information set:

1. Clearly, in the equation for the cyclical component of output we have an example of an ("ad hoc") assumption which is made in order to allow for persistent effects of forecast errors, namely, the assumption according to which the cyclical component of supply depends on its value in the previous period (with $\lambda < 1$). This, as underlined for example by Modigliani (1977), represents an "ad hoc" assumption which is necessary in order to make persistent the effects of uncorrelated forecast errors of the kind $p_t(z) - E[p_t|I_t(z)]$. The following passage of Sargent's book makes evident the "ad hoc" nature of the introduction of the term λy_{t-1}^C in the right-hand side of equation (2) which in Sargent's book is equation (9):

"... if $\lambda > 0$, the effects of forecast errors that are themselves serially uncorrelated will persist and make y_t serially correlated. Since it is a fact that y_t is strongly serially correlated, it is necessary to permit λ to exceed zero (and by a healthy amount) if the facts are to be accounted for in the context of (9)"

(Sargent, 1979, p.329; my emphasis).

$$I_t(z) = (p_t(z), I_{t-1}) \quad (3),$$

where I_{t-1} , which is common to all markets, contains all other information such as past course of demand shifts, output and prices.

With regard to the current price in market z , which is observed by individuals in that market, it is assumed that:

$$p_t(z) = p_t + u_t(z) \quad (4),$$

where p_t is the current general price and $u_t(z)$, the deviation of the price in market z from the general price, deviation which is due to relative demand shocks and is such that $u_t(z) \sim N(0, \sigma_u^2)$.

Although individuals do not observe the general price level, i.e., $p_t \notin I_t(z)$, it is assumed that they, on the basis of past information contained in I_{t-1} , know p_t to be normally distributed with mean \hat{p}_t and constance variance σ^2 .

Thus, we can write:

$$p_t = E[p_t | I_{t-1}] + \xi_t = \hat{p}_t + \xi_t \quad (5),$$

where $\xi_t \sim N(0, \sigma^2)$.

Taking account of the partition of the information set, and employing the recursive regression formula¹, we obtain:

1. See Sargent, 1979, pp.203-213, 324-331.

$$\begin{aligned} E[p_t | I_t(z)] &= E[p_t | (p_t(z), I_{t-1})] = \\ &= E[p_t | I_{t-1}] + E[(p_t - E[p_t | I_{t-1}]) | (p_t(z) - E[p_t(z) | I_{t-1}])] \quad (6). \end{aligned}$$

From (4) and (5), we have:

$$p_t - E[p_t | I_{t-1}] = \xi_t,$$

and

$$p_t(z) - E[p_t(z) | I_{t-1}] = \xi_t + u_t,$$

where, in addition to $E u_t = 0$, $E u_t^2 = \sigma_u^2$, $E \xi_t = 0$, $E \xi_t^2 = \sigma^2$, for all t , it is assumed that $E u_t u_{t'} = E \xi_t \xi_{t'} = 0$, for all $t \neq t'$, and $E \xi_t u_t = 0$ for all t .

Thus, inserting in (6), we obtain:

$$\begin{aligned} E[p_t | I_t(z)] &= \hat{p}_t + \frac{\text{Cov}[\xi_t, (\xi_t + u_t)]}{\text{Var}(\xi_t + u_t)} [p_t(z) - \hat{p}_t] = \\ &= \hat{p}_t + \frac{\sigma^2}{\sigma^2 + \sigma_u^2} [p_t(z) - \hat{p}_t] = \\ &= \theta \hat{p}_t + (1 - \theta) p_t(z) \quad (7), \end{aligned}$$

where $\theta = \sigma^2 / (\sigma_u^2 + \sigma^2)$ is the fraction of total variance due to relative price variation.

Finally, inserting (7) in (2) and then (2) in (1), we obtain the following expression for supply in market z at time t :

$$y_t(z) = b\theta[p_t(z) - \hat{p}_t] + \lambda y_{t-1}^c(z) + y_t^p \quad (8).$$

To close this Lucasian model, we specify an aggregate demand function which, in log terms, is of the form:

$$y_t + p_t = x_t \quad (9),$$

where, on the left-hand side, we have the (log of) nominal income whereas on the right-hand side an exogenous shift variable¹. The exogenous shift variable is not immediately observable but it is assumed that individuals know it to be normally distributed with:

$$E(\Delta x_t) = \delta \text{ and } \text{Var}(\Delta x_t) = \delta_x^2,$$

where $\Delta x_t = x_t - x_{t-1}$.

Averaging over markets, from (8) we obtain the following expression for the total supply of the economy:

$$\begin{aligned} y_t &= y_t^p + b\theta(p_t - \hat{p}_t) + \lambda y_{t-1}^c = \\ &= y_t^p + b\theta(p_t - \hat{p}_t) + \lambda(y_{t-1} - y_{t-1}^p) \end{aligned} \quad (10).$$

1. See Lucas, 1973, p.328.

2. This formulation implies that, in this model, nominal income rather than money is used as the "forcing variable". See Lucas, 1981, p.12.

Thus, assuming that the permanent component of supply follows a trend line, for example such that $y_t^p = \beta t$, we obtain the following equilibrium condition:

$$y_t^p + b\theta(p_t - \hat{p}_t) + \lambda(y_{t-1} - y_t^p + \beta) + p_t - x_t = 0 \quad (11),$$

from which, as we shall see in the next Section, the equilibrium solution of the model can be easily obtained.

2.1.2.1. The Equilibrium Solution of the Model

In the foregoing analysis of Barro's model we have not considered the procedure employed to obtain the (final) equilibrium solutions in output and price. Given that this procedure represents the solution method that is always used in the equilibrium approach to business cycle theory, however, it may be interesting and enlightening to analyse it in greater detail for the model we are now considering¹.

As we have above already stressed, this method consists, first, in "guessing" a solution for the price level in terms of unknown coefficients on the relevant exogenous variables and, then, in

1. Lucas is the author who, adapting the method from his 1972 article on "Expectations and the Neutrality of Money", where in turn the method was based on the ideas of Muth, introduced this method for the solution of an equilibrium model in his 1973 paper. See Lucas, 1973, p.329f.

using the market clearing condition, which is assumed to be always satisfied, to determine the values of the unknown coefficients.

For Lucas' model which we are now considering, this kind of "trial" solution must be of the form:

$$p_t = a_0 + a_1 x_t + a_2 x_{t-1} + \dots \\ \dots + b_1 y_{t-1} + b_2 y_{t-2} + \dots + c_0 y_t^p \quad (12),$$

where the right-hand side contains all variables that may influence the current general price.

Given this trial price solution, we also have:

$$E[p_t | I_{t-1}] = \hat{p}_t = a_0 + a_1 E[(\Delta x_t + x_{t-1}) | I_{t-1}] + \\ + a_2 x_{t-1} + \dots + b_1 y_{t-1} + b_2 y_{t-2} + \dots \\ \dots + c_0 y_t^p = a_0 + a_1 (\delta + x_{t-1}) + a_2 x_{t-2} + \dots \\ \dots + b_1 y_{t-1} + b_2 y_{t-2} + \dots + c_0 y_t^p \quad (13).$$

Thus, in order condition (11) to hold, we must have:

$$y_t^p + b\theta[a_1 x_t - a_1(\delta + x_{t-1})] + \lambda(y_{t-1} - y_t^p + \beta) + \\ + (a_0 + a_1 x_t + a_2 x_{t-1} + \dots + b_1 y_{t-1} + b_2 y_{t-2} + \dots \\ \dots + c_0 y_t^p) - x_t = 0 \quad (14).$$

For this identity in x_{t-i} , y_{t-j} ($i=0,1,2,\dots; j=0,1,2,\dots$) and y_t^p to hold, all coefficients must be equal to zero, i.e., we must have:

$$\text{Coeff}(x_t) = 0, \text{Coeff}(x_{t-1}) = 0,$$

$$\text{Coeff}(y_{t-1}) = 0,$$

$$\text{Coeff}(y_t^p) = 0,$$

$$-b_1 a_1 + \lambda \beta + a_0 = 0,$$

$$a_3 = a_4 = \dots = b_2 = b_3 = \dots = 0,$$

from which we find:

$$a_0 = \frac{b\theta\delta}{(1+b\theta)} - \beta\lambda,$$

$$a_1 = \frac{1}{(1+b\theta)},$$

$$a_2 = \frac{b\theta}{(1+b\theta)},$$

$$b_1 = -\lambda,$$

$$c_0 = \lambda - 1.$$

Inserting these values of the coefficients in (12), and then inserting the equilibrium value of p_t in (11), we obtain the following equilibrium solution¹ for the cyclical component of output :

$$\begin{aligned} y_t^c &= x_t - p_t - y_t^p = \\ &= -\delta\pi + \beta\lambda + \pi\Delta x_t - (\lambda - 1)y_t^p + \lambda y_{t-1} = \end{aligned}$$

1. See Lucas, 1973, pp.329.

$$= -\delta\pi + \pi\Delta x_t + \lambda y_{t-1}^c,$$

or:

$$\begin{aligned} y_t^c &= \frac{\pi}{1 - \lambda L} (\Delta x_t - \delta) = \\ &= \pi \sum_{j=0}^{\infty} (\lambda L)^j (\Delta x_t - \delta) = \\ &= \pi \sum_{j=0}^{\infty} \lambda^j (\Delta x_{t-j} - \delta) \end{aligned} \quad (15),$$

where $\pi = b\theta/(1 + b\theta)$ and where L is the lag operator such that $L^j x_t = x_{t-j}$.

The result we have obtained in (15) shows that the cyclical component of the equilibrium solution in y is a weighted sum of actual and past unperceived parts of shifts in demand that are exogenously given in the model. Thus, notwithstanding the impressive effort made by Lucas, the dynamics of the economy in his model is ultimately exogenous. This problem will be discussed in greater length at the end of this chapter. For the moment, it is enough to stress that this reliance on exogenous shocks has often been considered an important shortcoming in this kind of models. Bearing this in mind, it is much easier to understand recent

1. To obtain this expression for the cyclical component of output we have used the assumption according to which the permanent component of output follows the trend line $y_t^p = \beta t$. From this assumption, indeed it follows that:

$$\beta = y_t^p - y_{t-1}^p.$$

contributions which, while maintaining all other assumptions of the "standard" equilibrium model, attempt to explain endogenously the business cycle or, at least, without relying only on exogenous, monetary shocks¹.

2.1.3. Imperfect Information and Random Disturbances: Concluding Remarks

To sum up, the two problems we have analysed in the previous Section, namely the exogenous dynamics of the system and the "ad hoc" specification of the information set may well account for the recent developments in equilibrium business cycle theory. Indeed, the main purpose of the latter can be seen to be that of showing that the existence of fluctuations in an equilibrium model does not necessarily depend on exogenous shocks or special specification of the information set; in other words, to give content to Hayek's famous programme according to which:

1. See Grandmont, 1985, pp.996-997. With regard to this problem, Long-Plosser's purpose is clearly stated: in their opinion, it is useful in an equilibrium business cycle model, in order to understand the relative importance of factors such as monetary disturbances, to deliberately ignore them. This attempt to assess the relative importance of other factors seems to reveal, even if this is not explicitly stated, a certain embarrassment with regard to the results obtained in "standard" equilibrium models. Indeed, in the latter, as we have seen, there is a unique type of shock that accounts for all dynamics within the economy, namely, monetary shocks in Barro's model and shocks to nominal income in Lucas' model. For a criticism of this aspect of equilibrium models, see Vercelli, 1984, pp.154-155.

"...the incorporation of cyclical phenomena into the system of economic equilibrium theory, with which they are in apparent contradiction, remains the crucial problem of Trade Cycle Theory"

(Hayek, 1933, p.33f)¹

2.2. Equilibrium Models of the Business Cycle with Perfect Foresight and Absence of Random Monetary Disturbances

1. This is the passage in Hayek's book quoted by Lucas (1977). However, another passage in the same book seems to be more useful in helping us to fully understand the most recent contributions to equilibrium business cycle theory and their attempt to reduce the reliance of equilibrium models on random shocks. Indeed, in the discussion of contemporary monetary theories, Hayek classifies them among the exogenous theories

"...for they begin with arbitrary interference on the part of the banks" and conclude that "...This is, perhaps, one of the main reasons for the prevailing scepticism concerning the value of such theories. A theory which has to call upon the deus ex machina of a false step by bankers, in order to reach its conclusion is, perhaps, inevitably suspect"

(Hayek, 1933, pp.144-145)

However, in Hayek's opinion, for this kind of theory,

"...it is quite unnecessary to adduce interference on the part of the banks in order to bring about a situation of alternating boom and crisis. By disregarding those divergencies between the natural and money rate of interest which arise automatically in the course of economic development, and by emphasizing those caused by an artificial lowering of the money rate, the Monetary Theory of the Trade Cycle deprives itself of one of its strongest arguments; namely, the fact that the process which is described must always recur under the existing credit organization, and that it thus represents a tendency inherent in the economic system, and is in the fullest sense of the word an endogenous theory"

(ib., pp.146-147; original emphasis).

2.2.1. Grandmont's Paper

To understand the relevance of Grandmont's criticism of the standard equilibrium business cycle model, it is useful to consider the model he uses to show that, in contrast to currently accepted views, a monetary competitive economy may undergo deterministic" fluctuations under laissez faire¹.

The model adopted by Grandmont to show this is an overlapping generations model where agents live only two periods and are identical²; the economy is not subject to any shock of any sort; markets clear at any date; and only one perishable consumption good is produced, without production lag, from the labor supplied by consumers. Moreover, the marginal productivity of labour is supposed to be equal to one so that, given also the other simplifying assumptions, profits maximization implies that the money prices of the good and of the labour are equal³.

1. See Grandmont, 1985, pp.996-997, where this purpose is expressed. The expression "deterministic" fluctuations is taken by Grandmont to mean fluctuations that are neither attributable to exogenous shocks nor to any variation of policy.

2. As is stressed by Grandmont himself, this simplifying assumption is equivalent to assuming that there is only one individual in each generation. From now on, we will refer to these as "the young" and "the old".

3. If MP_L is the marginal productivity of labour, which is assumed to be equal to one; W , the money wage rate; and p , the money price of the good, profits maximization requires:

For each agent, the decision problem that must be solved consists in choosing a level of consumption of good and of leisure so as to maximize utility and such that:

$$c_i \geq 0,$$

and

$$0 \leq \bar{l}_i - l_i \leq \bar{l}_i,$$

where c_i is the agent's consumption in period $i(=1,2)$; \bar{l}_i , his labour endowment in period i ; and l_i , his labour supply.

In particular, it is important to note that for the "young" individual living at time t , the problem to be solved involves expected magnitudes, in particular the expected value of the next period price (p^e). Indeed, his problem resides not only in choosing c_1 and $(\bar{l}_1 - l_1)$ for the current period but also in planing, on the basis of his expectations on next period price, c_2 and $(\bar{l}_2 - l_2)$.

(Footnote continued from previous page)

$$W_t/p_t = MP_r L_t,$$

from which:

$$W_t/p_t = 1 \iff W_t = p_t.$$

Thus, assuming that the "young" agent's intertemporal tastes are represented by the separable utility function $U_1(c_1, \bar{l}_1 - l_1) + U_2(c_2, \bar{l}_2 - l_2)$ and that he can hold a nonnegative money balance m , we can say that the decision to be taken by him is to choose $c_1 \geq 0$, $0 < l_1 < \bar{l}_1$, $m \geq 0$, and to plan $c_2 \geq 0$ and $0 \leq l_2 \leq \bar{l}_2$ so as to solve the following maximization problem:

$$\begin{aligned} \text{Max } & U_1(c_1, \bar{l}_1 - l_1) + U_2(c_2, \bar{l}_2 - l_2) \\ \text{s.t. } & pl_1 - pc_1 - m = 0, \\ & p^e l_2 + m - p^e c_2 = 0^1, \end{aligned}$$

where the first of the two constraints is the current budget constraint whereas the second is the expected budget constraint.

Under the assumptions that are made by Grandmont², this maximization problem has a unique solution. Writing $z_i(\theta)$ and $m^d(p, p^e)$, where $\theta = p/p^e$ ³, for the excess demands for the good and the demand for money which result from the intertemporal

1. The $U_i(\dots)$ ($i=1,2$) are assumed to be continuous, increasing in both arguments and strictly concave.

2. See Grandmont, 1985, p.999.

3. Given the definition of θ , we can say that:

$$\theta - 1 = \frac{p - p^e}{p^e},$$

is the consumer's expected real interest rate. See Grandmont, 1985, p.999.

utility maximization, and inserting in the budget constraints, we obtain the following identities:

$$pz_1(\theta) + m^d(p, p^e) \equiv 0 \quad (1),$$

and

$$p^e z_2(\theta) \equiv m^d(p, p^e) \quad (2),$$

which imply:

$$\theta z_1(\theta) + z_2(\theta) \equiv 0 \quad (3).$$

The "young" agent problem we have analysed can be seen to comprise two distinct sub-problems¹.

Indeed, the two budget constraints can be written as:

$$p[c_1 + (\bar{l}_1 - l_1)] + m = p\bar{l}_1 = pa_1 + m \quad (4),$$

$$p^e[c_2 + (\bar{l}_2 - l_2)] = m + \bar{l}_2 p^e = p_2^e a_2 \quad (5),$$

where $a_i = c_i + (\bar{l}_i - l_i)$ represents the "real wealth", equal to $(\bar{l}_1 - m/p)$ when $i = 1$ and to $(\bar{l}_2 + m/p^e)$ when $i = 2$.

Thus, the "young" agent can solve his maximization problem in the following two steps. First, given $a_i > 0$, he chooses c_i and $(\bar{l}_i$

1. See Grandmont, 1985, p.1000.

$-l_i$) so as to maximize $U_i(...)$ subject either to $pa_1 + m = p\bar{l}_1$, when $i=1$, or to $p^e a_2 = m + p^e \bar{l}_2$, when $i=2$; the results of this maximization are the optimum values $c_i(a_i)$ and $l_i(a_i)$ such that $c_i(a_i) + [\bar{l}_i - l_i(a_i)] = a_i$ and $U(c_i(a_i), [\bar{l}_i - l_i(a_i)]) = V_i(a_i)$. Second, he chooses $m > 0$, $a_1 > 0$ and $a_2 > 0$ so as to maximize $V_1(a_1) + V_2(a_2)$ subject to $pa_1 + m = p\bar{l}_1$ and $p^e a_2 = m + p^e \bar{l}_2$; the results of this maximization are optimum values of $a_1 - \bar{l}_1 = c_1 - l_1$, $a_2 - \bar{l}_2 = c_2 - l_2$, and m , that coincide with $z_1(\theta)$, $z_2(\theta)$, and $m^d(p, p^e)$ respectively.

The purpose in splitting the original utility maximization problem into the two sub-problems is to allow us to concentrate our attention on the indirect utility functions $V_i(a_i)$ ¹ rather than on the original functions $U_i(...)$. In particular, it is crucial to concentrate our attention on the relation between the $V_i(a_i)$ functions and the excess demand functions $z_i(\theta)$.

Writing $\bar{\theta}$ for the inverse of the marginal rate of substitution at the endowment point, i.e. $\bar{\theta} = V'_1(\bar{l}_1)/V'_2(\bar{l}_2)$, it is possible to show² that the excess demands are both equal to zero for $\theta \leq \bar{\theta}$, whereas for $\theta > \bar{\theta}$, we have:

$$-\bar{l}_1 < z_1(\theta) < 0, \text{ and } z_2(\theta) > 0.$$

1. Concerning these indirect utility functions a stronger assumption is made by Grandmont according to which they are continuous on $[0, +\infty)$ and twice differentiable on $(0, +\infty)$, with $V'_i(a_i) > 0$, $V''_i(a_i) < 0$, and $\lim_{a_i \rightarrow 0} V'_i(a_i) = +\infty$.

2. See Grandmont, 1985, p.1000, Lemma 1.1.

Moreover, for every $\theta \geq \bar{\theta}$, the following equality must hold:

$$V'_1(\bar{l}_1 + z_1(\theta)) = \theta V'_2(\bar{l}_2 + z_2(\theta)) \quad (6),$$

or, equivalently:

$$-z_1(\theta)V'_1(\bar{l}_1 + z_1(\theta)) = z_2(\theta)V'_2(\bar{l}_2 + z_2(\theta))^1 \quad (7).$$

The important implication of these results is that the "young" agent has a positive demand for money if and only if $\theta > \bar{\theta}$, where, to ensure that there are enough incentives to save even when the price of the good is expected to be constant, and therefore $\theta = 1$, it is assumed that $\bar{\theta} < 1^2$.

Restricting attention to the interval $(\bar{\theta}, +\infty)$, where the excess demand functions are both different from zero, it is possible to

1. It is easy to show that (6) and (7) are equivalent. Indeed, from (3), we obtain:

$$\theta = -z_2(\theta)/z_1(\theta),$$

from which, inserting in (6), we obtain (7).

2. See Grandmont, 1985, p.1001.

express $z'_1(\theta)$ and $z'_2(\theta)$ in terms of $V'_i(a_i)$ and $V''_i(a_i)$ ¹. On the basis of these expressions, however, it is possible to conclude that $z'_2(\theta) > 0$, for all $\theta \geq \bar{\theta}$, whereas something of an analogous nature cannot be said with regard to the sign of $z'_1(\theta)$ ².

1. Indeed, differentiating with respect to θ both sides of (7) and of (3), we obtain the following system:

$$\begin{bmatrix} (V'_1 + z_1 V''_1) & (V'_2 + z_2 V''_2) \\ \theta & 1 \end{bmatrix} \begin{bmatrix} z'_1 \\ z'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -z_1 \end{bmatrix},$$

from which:

$$\begin{bmatrix} z'_1 \\ z'_2 \end{bmatrix} = \frac{1}{z_1(V''_1 + \theta^2 V''_2)} \begin{bmatrix} 1 & -V'_2 - z_2 V''_2 \\ -\theta & V'_1 + z_1 V''_1 \end{bmatrix} \begin{bmatrix} 0 \\ -z_1 \end{bmatrix},$$

i.e.:

$$z'_1(\theta) = \frac{V'_2(\bar{l}_2 + z_2(\theta)) + z_2(\theta)V''_2(\bar{l}_2 + z_2(\theta))}{V'_1(\bar{l}_1 + z_1(\theta)) + \theta^2 V''_2(\bar{l}_2 + z_2(\theta))},$$

$$z'_2(\theta) = - \frac{V'_1(\bar{l}_1 + z_1(\theta)) + z_1(\theta)V''_1(\bar{l}_1 + z_1(\theta))}{V'_1(\bar{l}_1 + z_1(\theta)) + \theta^2 V''_2(\bar{l}_2 + z_2(\theta))}.$$

Thus, we see that, assuming that the indirect utility functions are twice differentiable, it is possible to express $z'_1(\theta)$ and $z'_2(\theta)$ in terms of the first and second derivatives of those functions.

2. Indeed, given that $V'_1 > 0$, $z_1(\theta) < 0$, and $V''_1 < 0$, the numerator of the expression for $z'_2(\theta)$ we have obtained in the previous footnote is surely positive whereas the denominator is surely negative. On the contrary, given that $V'_2 > 0$, $z_2(\theta) < 0$, $V''_2 < 0$, the numerator of the expression for $z'_1(\theta)$ can be either positive or negative.

The problem is that a change in $\theta = p/p^e$ entails changes in the excess demand functions both because of an intertemporal substitution effect and of an income effect¹. The total effect on $z_2(\theta)$ of an increase in θ is undoubtedly positive because both effects work in the same direction, whereas it can be either positive or negative on $z_1(\theta)$. Indeed, an increase in θ , on the one hand, entails a decrease in $z_1(\theta)$ because of the substitution effect and, on the other hand, an increase in $z_1(\theta)$ because of the income effect. Thus, although $z'_1(\bar{\theta}) < 0$ ² and hence $z'_1(\theta) < 0$ for larger but close to $\bar{\theta}$, nothing can be said for large values of θ .

This means that there is a "potential conflict" between intertemporal substitution and income effects and this "potential conflict" is seen by Grandmont as the origin of the business cycle analysed in his paper³.

To study the dynamics of the economy, it is necessary to close the model by considering, besides the "young" agent problem we have analysed, also the decision problem facing the "old" agent living at t .

1. The income effect of a change in θ arises because the "intertemporal income", which can be obtained by summing the two income constraints of the second of the two subproblems, is:

$$\theta a_1 + a_2 = \theta \bar{l}_1 + \bar{l}_2.$$

2. This is true because $\bar{\theta} z_1(\bar{\theta}) + z_2(\bar{\theta}) = 0$ so that $z'_1(\bar{\theta}) = -z'_2(\bar{\theta})/\bar{\theta}$.

3. See Grandmont, 1985, p.1001.

For the "old" agent, the problem is to maximize utility taking account of the fact that his excess demand at any date t cannot exceed the real value of his money stock M . Thus, the maximization problem he must solve is the following:

$$\text{Max } U(c, \bar{l} - 1)$$

$$\text{s.t. } c \geq 0:$$

$$0 \leq \bar{l} - 1 \leq \bar{l},$$

$$p_l + M - p_c = 0,$$

where M is the outstanding stock of money.

As for the "young" agent, the results of this maximization will be optimal values of the type $c(a)$ and $l(a)$ where $a = c + (\bar{l} - 1) = \bar{l} + M/p$, such that $c(a) - l(a) = M/p$.

Thus, we obtain the following market clearing conditions that must be satisfied at time t :

$$z_1(p_t/p_{t+1}^e) + M/p_t = 0 \quad (8),$$

$$m^d(p_t, p_{t+1}^e) = M \quad (9).$$

A crucial aspect in such an overlapping generations model where individuals are assumed to live for more than one period is, as we have seen, that the "young" must solve an intertemporal utility maximization problem. For this reason, in order to derive the dynamics of the system from the microeconomic behaviours, it is crucial to specify the mechanism by means of which the "young" agent living at t computes the expected price for the following

period. This is equivalent to specifying an expectation function, i.e., a function ψ relating the expected price to present and past information held by the "young" agent on the structure of the economy¹.

According to the specification chosen by Grandmont², the information held by the "young" agent about the past is composed only of current and past prices; thus, we can write:

$$p_{t+1}^e = \psi(p_t, \dots, p_{t-T}) \quad (10),$$

and:

$$\begin{aligned} z_1(p_t / \psi(.)), \\ m^d(p_t, \psi(.)), \end{aligned}$$

for the results of the "young" agent intertemporal utility maximization.

Thus, (8) and (9) become³:

1. The importance of the specification of the expectation function is strongly emphasised by Fitoussi. See Fitoussi, 1983, pp.9-10.

2. See Grandmont, 1985, p.1008.

3. The two conditions are clearly equivalent by Walras's law. Indeed, from (1), we have:

$$m^d(p_t, p_{t+1}^e) = -p_t z_1(p_t / p_{t+1}^e).$$

$$z_1(p_t / \psi(.)) + M/p_t = 0, \quad \text{for the goods market} \quad (11),$$

$$m^d(p_t, \psi(.)) = M, \quad \text{for the money market} \quad (12).$$

As underlined by Grandmont¹, the existence of a solution p_t to this system is guaranteed if there is enough intertemporal substitution when p_t varies, that is if the elasticity of $\psi(.)$ with respect to the current price is between 0 and 1:

$$0 < [\partial \psi(.) / \partial p_t] / [\psi(.) / p_t] < 1,$$

a condition which is assumed to hold by Grandmont². In this case, the intertemporal substitution effect is likely to reinforce the real balance effect of a change in the current price and may be strong enough to equilibrate the market. Moreover, in this case, the market clearing conditions allow for a unique solution of the type:

$$p_t = W(p_{t-1}, \dots, p_{t-T}) \quad (13).$$

1. See Grandmont, 1985, p.1008.

2. It is assumed, moreover, that:

$$\lim_{p_t \rightarrow \infty} \psi(.) / p_t = 0.$$

See Grandmont, 1985, p.1008. See also Fitoussi, 1983, p.9.

It is important at this point to notice that the expectation function implies an assumption about the "learning process" characterizing the "young" individual. However, in the case in which the market clearing conditions have an infinite sequence of solutions (p_t) such that $p_{t+k} = p_t$ for all t , i.e., in the case of a periodic competitive equilibrium, the learning process plays no role. Indeed, in this case, although Grandmont's analysis is wholly developed in terms of the expectation function, the mechanism by means of which expectations are formed, and possibly revised, plays no role because the "young" individual is supposed to possess perfect foresight. Indeed, it is assumed that the expectation function is consistent with periodicity k , so that¹:

$$\psi(p_t, \dots, p_{t-T}) = p_{t-k+1} \quad (14),$$

i.e., $p_{t+1}^e = p_{t+1}^2$.

Given this assumption, (12) becomes:

$$m^d(p_t, p_{t+1}) = p_{t+1} z_2(\theta_t) = M \quad (15),$$

1. See Grandmont, 1985, p.1010.

2. The rationale for this assumption, in Grandmont's opinion, is the fact that

"...agents 'learn' and thus make mistakes when their environment is chaotic, but that they make correct forecasts along any solution...that has period k "

(Grandmont, 1985, p.1010).

so that, inserting in the condition for the goods market, we obtain:

$$z_1(\theta_t) + z_2(\theta_{t-1}) = 0 \quad (16),$$

where $\theta_t = p_t/p_{t+1}$ such that $\bar{\theta} < \theta_t < \bar{\bar{\theta}} = z_2^{-1}(\bar{l}_2)^1$.

In the interval $[\bar{\theta}, +\infty)$, $z_2(\theta)$ is always increasing so that we can invert it and write:

$$\theta_{t-1} = z_2^{-1}(-z_1(\theta_t)) = \phi(\theta_t) \quad (17),$$

where $\phi : [\bar{\theta}, +\infty) \longrightarrow [\bar{\theta}, \bar{\bar{\theta}})$.

As underlined by Grandmont, although the equivalence concerns only periodic competitive equilibria, to find a periodic competitive equilibrium with perfect foresight is equivalent to find a periodic solution of the "backward" difference equation (17)².

1. This restriction on the values of θ_t is explained in the following way: we have $M > 0$ and hence $z_2(\theta_t) \geq 0$ so that $\theta_t > \bar{\theta}$.

Moreover, one has $z_2(\theta_t) = -z_1(\theta_{t+1}) < \bar{l}_1$ so that $\theta_t < z_2^{-1}(\bar{l}_1) = \bar{\bar{\theta}}$.

2. Indeed, if (p_1^*, \dots, p_k^*) is a periodic competitive equilibrium with period k , we can define $\theta_j^* = p_j^*/p_{j+1}^*$ ($j = 1, \dots, k$; $p_{k+1}^* = p_1^*$).

(Footnote continues on next page)

Then, it is possible to show how a cycle of period k is described by this equation by means of a graphic illustration.

To this end, Grandmont introduces the concept of a "trader's offer curve", i.e., the locus of all points of coordinates:

$$a_1 = \bar{l}_1 + z_1(\theta), \quad a_2 = \bar{l}_2 + z_2(\theta),$$

when θ varies.

To obtain the graph of this curve, however, we must distinguish two different cases because the slope of the offer curve depends on the signs of $z'_1(\theta)$ and $z'_2(\theta)$.

(Footnote continued from previous page)

In this case, we have $\theta_j^* > \bar{\theta}(\psi_j)$ and $(\theta_k^*, \dots, \theta_1^*)$ is a periodic orbit of ψ with period k . Conversely, if $(\theta_k^*, \dots, \theta_1^*)$ is a periodic orbit of ψ with period k , and if $\theta_j^* > \bar{\theta}(j)$, then, from $p_{t+1}z_2(\theta_t) = M$, we can obtain a sequence (p_1^*, \dots, p_k^*) , where $p_j^* = M/z_2(\theta_{j-1})$ such that $(\theta_0^* = \theta_k^*)$, which determines a periodic competitive equilibrium with period k . Incidentally, given that $z'_2(\theta) > 0$ ($\theta > \bar{\theta}$), this also shows that the prices of a periodic competitive equilibrium must all differ. See Grandmont, 1985, p.1005 and 1005f.

As we have already noted, $z_2'(\theta)$ is positive for all $\theta > \bar{\theta}$, whereas the sign of $z_1'(\theta)$ is ambiguous. However, as stressed by Grandmont¹, an important role in this regard is played by what can be called "Arrow-Pratt relative degree of risk aversion" :

$$R_i(a_i) = -V_i''(a_i)a_i/V_i'(a_i).$$

where $i=1,2$.

Indeed, it is possible to show² that, for every $\theta \geq \bar{\theta}$, $z_1'(\theta) < 0$ if and only if $R_2(\bar{l}_2 + z_2(\theta)) < (\bar{l}_2 + z_2(\theta))/z_2(\theta)$. As a consequence, (i) if $R_2(a_2) \leq 1$ for all $a_2 > 0$, then $z_1'(\theta) < 0$ for every $\theta \geq \bar{\theta}$, whereas (ii) if $R_2(a_2) \geq 0$ for every $a_2 > 0$ and if there exists $a_2 > 0$ such that $R_2(a_2) > 1$, then there exists a unique $\theta^* > \bar{\theta}$ such that $z_1'(\theta) < 0$ for every $\bar{\theta} \leq \theta < \theta^*$, $z_1'(\theta^*) = 0$, and $z_1'(\theta) > 0$ for every $\theta > \theta^*$.

Taking account of this, it is easy to draw the consumer's offer curve for the two cases considered in (i) and (ii) as is shown in Figure 2.1.a,b. In both cases, the point A has coordinates (\bar{l}_1, \bar{l}_2) ; the point B, coordinates $(0, \bar{l}_1 + \bar{l}_2)$ whereas all points on the straight line AB represent all other combinations in between such that:

$$a_1 + a_2 = \bar{l}_1 + \bar{l}_2.$$

1. See Grandmont, 1985, p.1001.

2. See Grandmont, 1985, p.1002, Lemma 1.3.

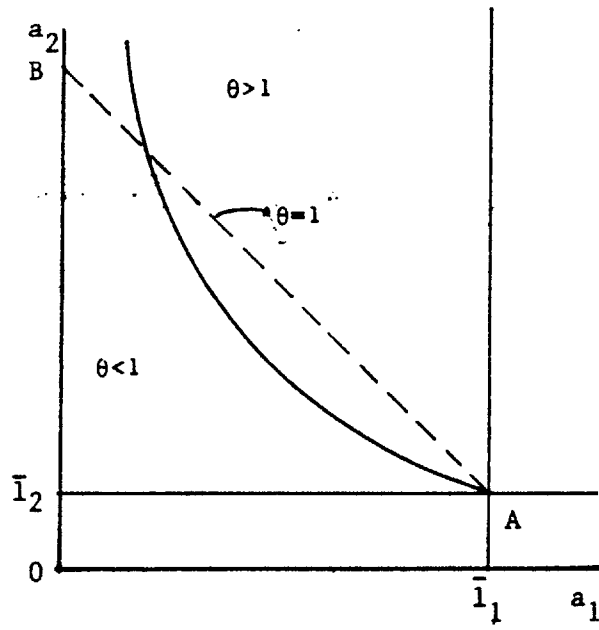


Fig.2.1.a

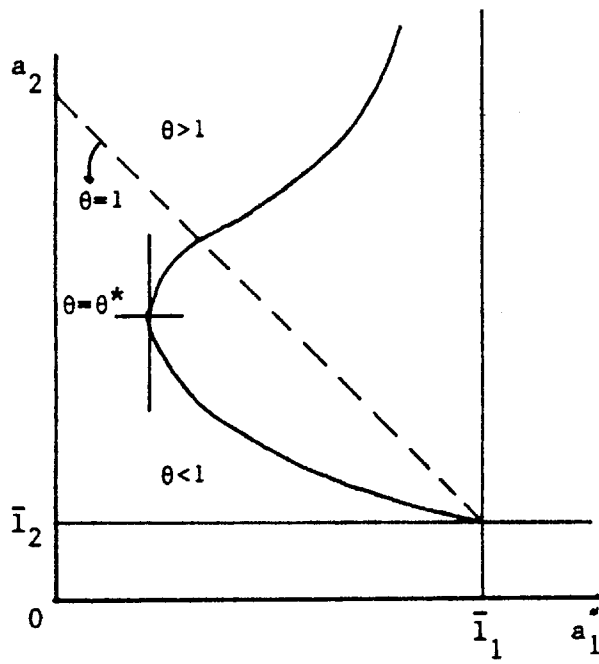


Fig.2.1.b

From the intertemporal budget line:

$$\theta(a_1 - \bar{l}_1) + (a_2 - \bar{l}_2) = 0,$$

we obtain:

$$\theta = - \frac{z_2(\theta)}{z_1(\theta)},$$

where, given that we are considering the case in which $\theta \geq \bar{\theta}$, we have:

$$z_2(\theta) > 0 \text{ and } z_1(\theta) < 0.$$

Thus:

$$\theta \geq 1 \iff z_1(\theta) + z_2(\theta) \geq 0 \iff a_1 + a_2 \geq \bar{l}_1 + \bar{l}_2,$$

and this means that the consumer's offer curve lies below the straight line AB when $\bar{\theta} < \theta < 1$ and above when $\theta > 1$.

To determine the slope of the offer curve, we note that, from $a_1 = l_1 + z_1(\theta)$, using the implicit function theorem, we obtain a relation between θ and a_1 ,

$$\theta = \theta(a_1),$$

such that:

$$\frac{d\theta}{da_1} = \theta'(a_1) = \frac{1}{z_1'(\theta)}.$$

Thus:

$$\begin{aligned} a_2 &= \bar{1}_2 + z_2(\theta) = \\ &= 1_2 + z_2(\theta(a_1)), \end{aligned}$$

is the required relation between a_1 and a_2 such that:

$$\frac{da_2}{da_1} = \frac{dz_2}{d\theta} \frac{d\theta}{da_1} = \frac{z_2'(\theta)}{z_1'(\theta)}.$$

Figure 2.1.a corresponds to case (i) of Grandmont's Lemma in which $z_1'(\theta)$ is negative for every $\theta \geq \bar{\theta}$ and hence the offer curve is a monotonically decreasing curve. On the other hand, Figure 2.1.b corresponds to case (ii) in which there exists a critical value θ^* such that the derivative of $z_1'(\theta)$, and consequently the slope of the consumer's offer curve, changes sign and is such that:

$$\frac{da_2}{da_1} \left\{ \begin{array}{l} < 0 \text{ for every } \bar{\theta} \leq \theta < \theta^*, \\ = 0 \text{ for } \theta = \theta^*, \\ > 0 \text{ for every } \theta > \theta^*. \end{array} \right.$$

In this case, drawing in the same graph both the trader's offer curve and the intertemporal budget constraint line, we obtain the representation of a cycle, for example of period $k = 2$, as is shown in Figure 2.2.

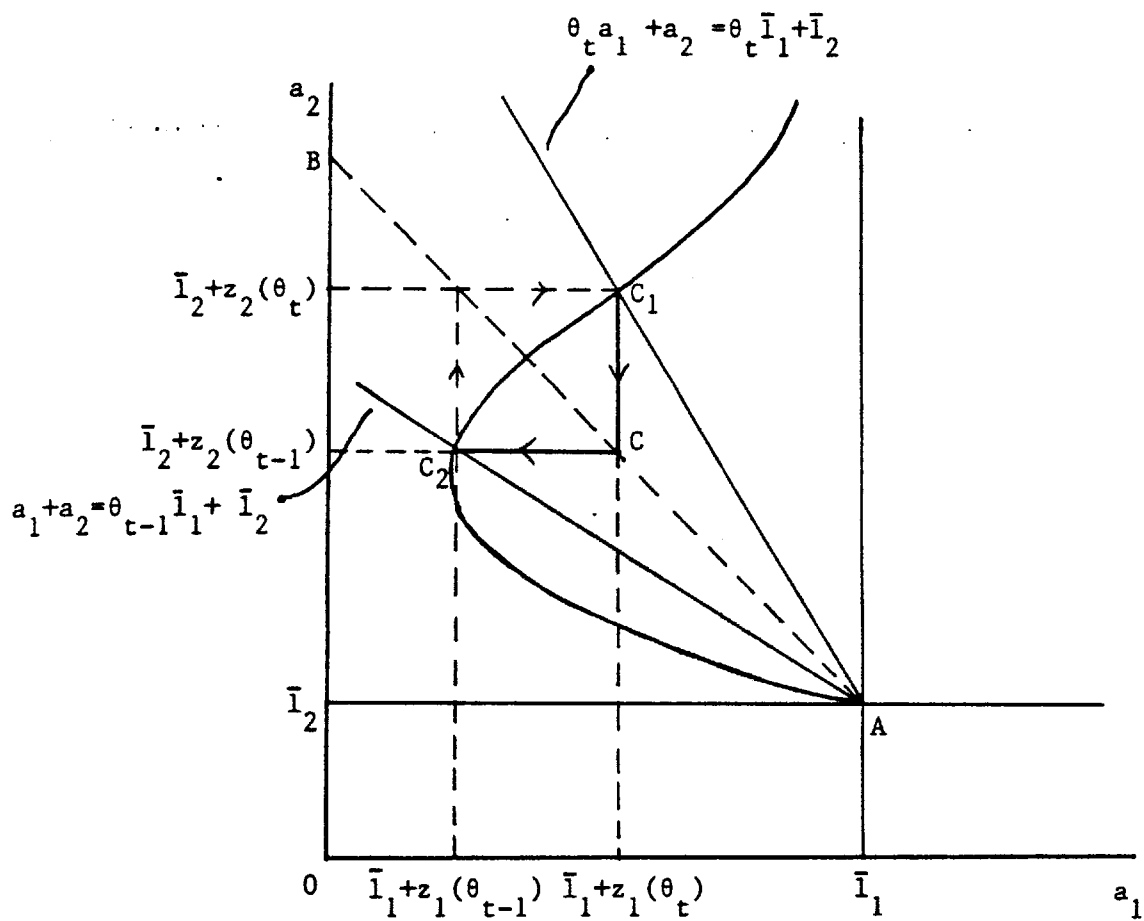


Fig.2.2

In such a framework, a periodic competitive equilibrium is identified with the orbit of the periodic sequence, i.e., with the k consecutive values the sequence takes, in the present case, with θ_2^* and θ_1^* . Thus, given the equivalence stated by Grandmont, the price periodic sequence, identifying a periodic competitive equilibrium, is (p_1^*, p_2^*) where $p_i^* = M/z_2(\theta_{i-1}^*)$ ($i=1,2$) with $\theta_0 = \theta_2^1$.

Clearly, the point is not to question the validity of Grandmont's statement that the orbits (θ_2^*, θ_1^*) and (p_1^*, p_2^*) are equivalent, in the case in which periodic competitive equilibria with perfect foresight are considered. The point, rather, is that it is quite an arduous task to interpret, from the economic theory point of view, conclusions reached on the basis of results obtained exploiting this equivalence. More precisely, the point is that it is difficult to understand what it means to study the dynamics of the system by means of a "backward" difference equation in an

1. Looking at Figure 2.2, it is easy to illustrate the backward dynamics associated with (16). To this end, it may be noted that the intersection of the consumer's offer curve with the intertemporal budget line for θ_t is the point C_1 with coordinates $(\bar{l}_1 + z_1(\theta_t), \bar{l}_2 + z_2(\theta_t))$. From C_1 , going vertically to the straight line AB , we obtain the point C where $a_1 = \bar{l}_1 + z_1(\theta_t)$ and $a_2 = (\bar{l}_2 + \bar{l}_2) - a_1 = \bar{l}_2 - z_2(\theta_{t-1})$. Then, going from C horizontally to the offer curve, we find a point C_2 of coordinates $(\bar{l}_1 + z_1(\theta_{t-1}), \bar{l}_2 + z_2(\theta_{t-1}))$. Finally, drawing through C_2 the corresponding intertemporal budget line, we determine the value of θ_{t-1} such that $\theta_{t-1} = \Phi(\theta_t)$.

overlapping generations model where the expectation of the "young" agent about the next period is (and must be) the basic ingredient. As stressed by Grandmont himself¹, this equation describes a (fictitious) backward perfect foresight dynamics whereas:

" ...after all, time should go forward in any meaningful dynamics"

(ib., p.1007; original emphasis)

However, although this is recognized by the author, the analysis is carried out by him in terms of the backward equation. Above all, underlying his choice there is a purely mathematical reason for which we cannot invert the excess demand function $z_1(\theta)$ even in the case in which the restriction of $z_1(\theta)$ to the interval $[\bar{\theta}, +\infty)$ is considered. On the contrary, the restriction of $z_2(\theta)$ to that interval is increasing; as a consequence, it has an inverse and the backward equation can be written².

The problem of the relation between the (fictitious) backward perfect foresight dynamics and the actual dynamics is touched upon

1. See Grandmont, 1985, p.1005.

2. The fact that equation (16) does not adequately define a "forward perfect foresight dynamics" is evident from Figure 2.2. Indeed, in that Figure, starting from θ_{t-1} , and following the procedure we have above described, one would get two possible points on the offer curve for which $z_2(\theta_t) + z_1(\theta_{t-1}) = 0$, namely the points C_1 and C_2 .

only in Section 3 of Grandmont's paper¹, where the author explicitly recognizes

" ...that the hypothesis of perfect foresight is not generally warranted out of 'long run' equilibria and that in order to study correctly the evolution of any economy, one should portray the traders as learning the dynamics of prices on the transition path"

(ib., p.1006).

In particular, a relationship between stability with learning (i.e., W-stability; see equation (13)) and stability under the perfect foresight backward dynamics (i.e., ϕ -stability; see equation (17)) is shown by Grandmont. However, this fact, rather than increasing one's interest in the analysis of the (fictitious) backward dynamics, should increase one's doubts about the relevance of the results that can be obtained in such a framework. After all, notwithstanding the impressive amount of highly refined (and very original, at least with regard to economic applications) mathematical techniques that are employed all through the paper in order to reach even the simplest conclusion, the model underlying the whole analysis is very simple, in our opinion too simple² to study such an important dynamic phenomenon as the persistent fluctuations in economic activity that have characterized, and

1. See Grandmont, 1985, pp.1005-1014.

2. An analogous opinion is expressed by Christopher Sims in his discussion of Grandmont's paper in Sonnenschein, 1986, p.37.

continue to characterize, the growth of capitalist economy¹.

In this connexion, a study of the specification chosen by Grandmont for the expectations function and of "manipulations" of it that are performed by him is of considerable value.

A self-evident problem with the specification of the expectation function is that it is "ad hoc" in the sense that is not derived from an optimization microeconomic behaviour. Although is not our intention to criticize business cycle models for their lack of microeconomic foundations, the way in which Grandmont specifies the expectation function appears to be an important contradictory element given that his purpose is, among other things,

"...to get a better foundation upon which to build a sound Keynesian, or more precisely non-Walrasian, business cycle theory"

(Grandmont, 1985, p.1040)

1. Grandmont's himself seems to be aware of this fact. Indeed, he admits, at the beginning of his conclusions, that his model is "too rudimentary". In the next line, however, it is rather confusing to read that the model is "plausible". After all, it is hard to understand how a model that does not give any role to mechanisms determining the process of accumulation, can be considered "plausible" in the context of business cycle theory. Although the importance of the consideration of something like multiplier and accelerator effects is not denied by Grandmont (see Grandmont, 1985, p.1040; Grandmont-Malgrange, 1986, p.3), in his model there is nothing to resemble them and it is clear that it could not be otherwise given the assumptions of competitive behaviour and markets clearing. Incidentally, it is well known that, as shown by the difference between the first and the second version of Kalecki's business cycle theory, concentrating attention on factors determining investment decisions allows one to give an economic meaning to backward as opposite to forward dynamic equations. See Kalecki, 1935, 1954; Steindl, 1984. See also Folsom, 1974.

Then, it is important to note the crucial role played in the model by "manipulations" of the expectation function.

As we have seen, firstly, in order to guarantee the existence of a solution to the market clearing conditions system, it is assumed that the elasticity of the expectation function with respect to the current price is between zero and one. This means that the existence of an equilibrium requires that the expectation of the "young" agent is largely insensitive to current price. As has been pointed out by Fitoussi¹, this condition is unlikely to be fulfilled in an inflationary environment. Secondly, as we have already noted, in the case in which a periodic competitive equilibrium exists, it is assumed that the "young" trader has perfect foresight so that the mechanism by means of which expectations are formed and revised plays no role and the market clearing condition, from which the backward difference equation is obtained, can be written².

For this reason, Grandmont's criticism, according to which the most recent reformulation of the equilibrium approach is not satisfactory because it must rely on the "ad hoc" assumption that

1. See Fitoussi, 1983, p.10.

2. Some interesting comments on the assumptions regarding the expectation function in Grandmont's paper are made by M. Woodford in his discussion of Grandmont's paper published in Sonnenschein, 1986, pp.41-44. In particular, he shows that the assumptions made by Grandmont, although they might seem reasonable,

"... are not as weak as they seem"

(ib., p.41).

cycles are due to exogenous shocks, does not seem to be well founded: as the "standard" equilibrium business cycle model can be criticized because of the role played by the specification of the information set, Grandmont's model can be criticized because of the crucial role played by the ("ad hoc") specification and "manipulations" of the expectation function.

Something else seems to be needed for a serious criticism of equilibrium business cycle theory¹.

To understand this, it is also useful to take account of the results obtained by Long and Plosser (1983) in their attempt to build a "real" equilibrium business cycle model, i.e., a model in which monetary disturbances play no role.

2.2.2. Long-Plosser's Paper

Another example of an attempt to build equilibrium business cycle models with no role for monetary disturbances is given by real equilibrium theories of the business cycle, for example, by

1. In Sims's opinion, although Grandmont begins his paper by placing his model in opposition to those of the New Classical school, his model, from a truly Keynesian perspective,

"... carries much the same message as that of other New Classical model"

(Sims's comments in Sonnenschein, 1986, p.39).

Long-Plosser's paper (1983)¹. As we read in the Introduction to the paper, the purpose in deliberately ignoring monetary disturbances is to assess the importance of other factors in the explanation of business cycles.

In the model, the crucial role is played by the joint operation of two hypotheses, one concerning consumer preferences and the other concerning production possibilities, which, in the authors' opinion, are generally consistent with all business cycle theories of which they are aware.

According to the preference hypothesis, consumers, at given prices, want to "spread" any unanticipated wealth increment over both time and commodities.

The production possibilities hypothesis, on the other hand, assumes constant returns to scale, smooth substitutability of inputs, and strictly diminishing marginal productivity of any given input in any given employment. Moreover, it is assumed that the production possibilities are such that each commodity may have alternative uses² and that, during any period, exogenous random shocks influence the production transformations that have just been described.

1. A previous example of a "real" business cycle model is given by King and Plosser in "The Behavior of Money, Credit, and Prices in a Real Business Cycle", NBER, Working Paper Series, n.853, February 1982.

2. As stressed by the authors, all these assumptions imply that there is a large range of both intratemporal and intertemporal substitution opportunities. See Long-Plosser, 1983, p.41.

In the model, where it is assumed that all commodities are perishable, a single, infinitely living individual, with given initial resources, production possibilities and tastes, is considered who, at the beginning of each period, chooses a preferred consumption-production plan.

In justifying this specification of the model, the authors make reference to Lucas' suggestion according to which it is important to give

"... an explanation of business cycles, grounded in the general laws governing market economies, rather than in political or institutional characteristics specific to particular countries or periods"

(Lucas, 1977, p.10; original emphasis)

For this reason, Long and Plosser make a self-evident effort consisting in reducing the importance of other existing explanations of the business cycle which, in their opinion, are based on particular hypotheses.

In this context, for example, it is clear why monetary disturbances are ignored in Long-Plosser's paper. Moreover, it is clear that the assumption made by the authors according to which all commodities are perishable has not only the purpose of making the analysis simpler but also the purpose of reducing the importance of explanations of the business cycle based on stocks or inventories.

Assuming competitive behaviour, the problem that has to be solved by the individual is the maximization, subject to the availability of resources and production possibilities, of the

expected value of his utility (U) that, as viewed at time $t=0$, can be written as:

$$U = \sum_{t=0}^{+\infty} b^t u(C_t, Z_t) \quad (1),$$

where $C_t(N \times 1)$ is the vector of commodity consumption in period t ; Z_t , the amount of leisure time consumed at time t ; and b ($0 < b < 1$) a discount factor.

Taking into consideration the case of no joint production and technical change¹, the production possibilities are represented by the following vector-valued function:

$$Y_{t+1} = F(L_t, X_t; \lambda_{t+1}) \quad (2),$$

where $Y_{t+1}(N \times 1)$ is the vector of total production of commodities; $L_t(N \times 1)$, the vector of labour inputs; $X_t(N \times N)$, the matrix of commodity inputs; and $\lambda_{t+1}(N \times 1)$, a random vector such that the vectors in the sequence $\{\lambda_t\}$ are independent and identically distributed.

With regard to the resources constraints, we have:

1. In such a framework, the assumption of no joint production implies that the matrix of inputs is a $(N \times N)$ matrix where N is the number of commodities in the model economy. The assumption of no technical change, on the other hand, implies that the vectors in the stochastic process $\{\lambda_t\}$ are independent and identically distributed. See Long-Plosser, 1983, pp.44-45.

$$z_t + \sum_{i=0}^N L_{it} = H, \quad t=0,1,2, \dots \quad (3),$$

where H is the total time available per period.

Moreover, we must have:

$$C_{jt} + \sum_{i=0}^N X_{ijt} = Y_{jt}, \quad j=1,2, \dots; \quad t=0,1,2, \dots \quad (4),$$

where X_{ijt} is the quantity of commodity j allocated at time t to the production of commodity i.

The results of this maximization problem are equilibrium values of C_t , z_t , L_t , X_t , and of the relative prices at time t that are stationary functions of the "state vector" $S_t = (Y_t, \lambda_t)^1$.

Thus, the description of the intertemporal evolution of the equilibrium quantities and prices is obtained in a recursive manner. First, given the initial state of the economy - $S_0 = (Y_0,$

1. See Long-Plosser, 1983, pp.44-46. The relative prices may be expressed in terms of the individual marginal utility of current leisure and partial derivatives of his "current welfare function" which can be expressed as:

$$V(S_t) = \text{Max } E(U|S_t) = \text{Max } E\left[\sum_{s=t}^{+\infty} b^{s-t} u(C_s, Z_s) | S_t\right]$$

s.t. resources availability and production possibilities,

i.e.:

$$V(S_t) = E\left\{\sum_{s=t}^{+\infty} b^{s-t} u[C(S_s), Z(S_s)] | S_t\right\}.$$

λ_0) - the allocation rules and price formulae determine C_0 , X_0 , L_0 , Z_0 , and the relative prices at time 0. Then, given these results, the production function and the random vector λ_1 determine $S_1 = (Y_1, \lambda_1)$, and so on¹.

Although this is a description of a very general nature, the major conclusions are drawn by Long and Plosser on the basis of the analysis of a concrete, specific example which is obtained by making specific assumptions regarding the form of preferences and production possibilities.

With regard to the "one-period" utility, it is assumed that:

$$u(C_t, Z_t) = v_0 \log Z_t + \sum_{i=1}^N v_i \log C_{it} \quad (5),$$

where $v_0 > 0$ and $v_i \geq 0$ for $i=1, 2, \dots, N$.

With regard to the production possibilities, on the other hand, it is assumed that the production functions are of "Cobb-Douglas" type such that:

$$Y_{i,t+1} = \lambda_{i,t+1} L_{it}^{b_i} \prod_{j=1}^N X_{ijt}^{a_{ij}}, \quad i=1, 2, \dots, N \quad (6),$$

where, because of the assumption of constant returns to scale, we must have:

1. See Long-Plosser, 1983, p.46.

$$b_i, a_{ij} \geq 0, \text{ for all } i, j,$$

such that:

$$b_i + \sum_{j=1}^N a_{ij} = 1.$$

In this case, the problem is to maximize:

$$\begin{aligned} E(U|S_t) &= E\left[\sum_{s=t}^{+\infty} b^{s-t} u(C_s, Z_s) | S_t\right] = \\ &= E\left[\sum_{s=t}^{+\infty} b^{s-t} [v_0 \log \hat{Z}_s + \sum_{i=1}^N v_i \log C_{is}] | S_t\right], \end{aligned}$$

subject to the availability of resources and production possibilities¹.

Leaving aside for the moment the way in which the equilibrium values are obtained, it is more interesting to analyse the consequences that these equilibrium values have for the equilibrium dynamic behaviour of output.

Inserting in the production functions (6) the equilibrium values of L and X , which are obtained as solutions of the above mentioned maximization problem, and taking log of both sides, it is possible to show that we obtain the following equation describing the

1. See Long-Plosser, 1983, p.48.

dynamics of output¹:

$$y_{t+1} = Ay_t + k + \eta_{t+1} \quad (7),$$

where $y_{t+1}(Nx1)$ is the vector $\{\log y_{t+1}\}$; $A = \{a_{ij}\}_{(NxN)}$ ²; $k_{(Nx1)}$ is a vector of constants; and $\eta_{t+1} = \{\log \lambda_{t+1}\}_{(Nx1)}$.

To simplify, it is assumed that:

$$E\eta_t = 0, \text{ for all } t,$$

$$E\eta_t \eta'_s = \begin{cases} I, & \text{for } t=s, \\ 0, & \text{for } t \neq s. \end{cases}$$

Given that the vector of shocks in the sequence $\{\lambda_t\}$ are independent, the specification of the matrix A plays the crucial role in explaining any intertemporal link between deviations of outputs from their expected values. In fact, we can say that we can have such an intertemporal link only if the matrix A is specified properly³. Moreover, the stability of the system depends also on the (eigenvalues of the) matrix A . In particular, given the

1. See Long-Plosser, 1983, p.52.

2. The elements of the matrix A are the elasticities of commodity outputs with respect to commodity inputs. Indeed, we have:

$$\frac{\partial y_{i,t+1}}{\partial x_{ilt}} \frac{\partial x_{ilt}}{\partial y_{i,t+1}} = a_{ij}.$$

See Long-Plosser, 1983, p.39.

3. See Long-Plosser, 1983, pp.52-53.

assumptions that have been made about the elements of the matrix A, dynamic stability is guaranteed.

Both statement can easily be proved.

First of all, we know that the elements of the matrix A, which represent the elasticities of commodity outputs with respect to commodity inputs, are non-negative; thus:

$$A \geq 0.$$

Moreover, given the assumption of constant returns to scale, the elements in each row of A are such that their sum is less than one, i.e.:

$$\sum_{j=1}^N a_{ij} = 1 - b_i < 1.$$

As a consequence, the eigenvalues of A have modulus less than one and dynamic stability is guaranteed.

Secondly, it is possible to show that also any correlation in deviations of output from normal values depends on the matrix A. To this end, we decompose the value of y into the steady-state component, given by the unconditional mean

$$E(y_{t+1}),$$

and the deviation from steady-state component:

$$y_{t+1}^c = y_{t+1} - E(y_{t+1}).$$

From (7), we obtain:

$$(I - A \cdot L)y_{t+1} = k + \eta_{t+1},$$

where L is the lag operator such that, for any variable z , $L^j z_t = z_{t-j}$.

Thus, we can write:

$$\begin{aligned} y_{t+1} &= (I - A)^{-1}k + (I - A L)^{-1}\eta_{t+1} = \\ &= (I - A)^{-1}k + \sum_{i=0}^{+\infty} A^i \eta_{t+1-i} \end{aligned} \quad (8).$$

From (8), we obtain:

$$\begin{aligned} E(y_{t+1}) &= (I - A)^{-1}k + \sum_{i=0}^{+\infty} A^i E(\eta_{t+1-i}) = \\ &= (I - A)^{-1}k \end{aligned} \quad (9),$$

and:

$$\begin{aligned} y_{t+1}^c &= y_{t+1} - E(y_{t+1}) = \\ &= \sum_{i=0}^{+\infty} A^i \eta_{t+1-i} \end{aligned} \quad (10),$$

i.e., the cyclical component of output proves to be a weighted sum of present and past exogenous disturbances.

Thus, we can write:

$$\begin{aligned}
 E(y_t^c y_t^{c'}) &= E\left(\left(\sum_{i=0}^{+\infty} A^i \eta_{t-i}\right)\left(\sum_{i=0}^{+\infty} A'^i \eta'_{t-i}\right)\right) = \\
 &= E[(\eta_t + A\eta_{t-1} + A^2\eta_{t-2} + \dots)(\eta'_t + A'\eta'_{t-1} + A'^2\eta'_{t-2} + \dots)] \\
 &= E(\eta_t \eta'_t) + AA'E(\eta_{t-1} \eta'_{t-1}) + A^2A'^2E(\eta_{t-2} \eta'_{t-2}) + \dots = \\
 &= \sum_{i=0}^{+\infty} A^i A'^i
 \end{aligned} \tag{11},$$

and, in general:

$$\begin{aligned}
 E(y_t^c y_{t-s}^{c'}) &= E\left[\left(\sum_{i=0}^{+\infty} A^i \eta_{t-i}\right)\left(\sum_{i=0}^{+\infty} A'^i \eta'_{t-s-i}\right)\right] = \\
 &= E[(\eta_t + A\eta_{t-1} + A^2\eta_{t-2} + \dots)(\eta'_{t-s} + A'\eta'_{t-s-1} + A'^2\eta'_{t-s-2} + \dots)] \\
 &= A^s E(\eta_{t-s} \eta'_{t-s}) + A^{s+1} E(\eta_{t-s-1} \eta'_{t-s-1}) + \dots = \\
 &= A^s \sum_{i=0}^{+\infty} A^i A'^i = A^s E(y_t^c y_t^{c'})
 \end{aligned} \tag{12}.$$

These results are interpreted by Long and Plosser¹ in terms of Frisch's (1933) distinction between impulse and propagation mechanisms. On the basis of this distinction, it is possible to say that the random disturbances that influence production transformations are only "exterior impulses" that are propagated, both through time and commodities, thanks to the assumptions concerning the matrix A. As we have seen, first of all, the fact that A is not a null matrix ensures propagation over time.

1. See Long-Plosser, 1983, p.53.

Secondly, the assumption that many commodities have many alternative productive employments, i.e., that the matrix A is characterized by the presence of many columns full of positive elements, guarantees propagation through commodities.

Thus, in this sense, the importance given to the specification of the matrix A by Long and Plosser can easily be understood especially when we bear in mind Lucas-Sargent's counterargument to the criticism of the equilibrium models according to which these models are not satisfactory because they cannot account for the persistence of cyclical movements¹. In particular, the propagation mechanism analysed by Long and Plosser, which is related to the specification of production transformations, seems to offer a fourth (alternative) propagation mechanism to be added to the three mechanisms listed by Lucas and Sargent².

1. The counterargument to this kind of criticism is given by Lucas and Sargent (1979, p.313) in the following passage:

"...This criticism is fallacious because it fails to distinguish properly between sources of impulses and propagation mechanisms, a distinction stressed by Frisch (1933)...Even though the new classical theory implies that the forecast errors which are the aggregate demand impulses are serially uncorrelated, it is certainly logically possible that propagation mechanisms are at work that convert these impulses into serially correlated movements in real variables like output and employment".

2. The specification of an alternative propagation mechanism is a necessary step in Long-Plosser's paper where it is assumed that individuals have complete information. The reason for this is that the three mechanisms suggested by Lucas-Sargent (1979, p.313) apply only to models in which agents have imperfect information.

Thus, Long-Plosser's model gives us new insights into the problem of "impulse vs. propagation" which, at first sight, may seem interesting. However, to understand the extent to which these insights are relevant, it is important to note that there is an important difference between the underlying idea of the "impulse-propagation" mechanism considered by Frisch and the underlying idea of that considered by Long-Plosser. In fact, in Frisch's 1933 model, there is an endogenous mechanism which explains fluctuations in economic activity. Given the assumptions concerning the values of the coefficients, however, the equilibrium is dynamically stable so that, to avoid that fluctuations die out completely, exogenous shocks are added to the deterministic part of the model. As a result, the overall motion of the economy depends both upon the endogenous motion¹ and the shocks and upon their interaction. For example, although the first causes of the motion are exogenous shocks, in the absence of which fluctuations would die away, the resulting oscillatory mechanism is not identical to the pattern of the shocks². On the contrary, in Long-Plosser's paper the inherent motion and the "exterior impulse" do not interact; in fact, in their model there is no inherent motion at all and, in the absence of shocks, output is always equal to its steady state value. This means that, although Long-Plosser's purpose is to

1. For example, in Frisch's model, the length of the cycles and the tendency towards dampening are determined by the intrinsic structure of the swinging system. See Frisch, 1933, p.155.

2. Cfr. Blatt, 1983, pp.198-204.

explain business cycles by means of "certain very ordinary principles" regarding the behaviour of maximizing individuals, in their model the existence of a cyclical component of output has no explanation other than random shocks.

2.3. Conclusions

On the basis of the analysis that we have developed, it seems possible to conclude that, firstly, in the "standard" equilibrium business cycle model, given the assumption of equilibrium in the goods (and labour) market and the assumption according to which individuals have rational expectations, the only possibility that still remains for explaining fluctuations in economic activity is to assume that agents have imperfect current information and that random exogenous shocks influence the economic activity.

To overcome these shortcomings, equilibrium models have been built where no role is played by imperfect information and monetary disturbances. However, in our opinion, these attempts do not seem to lead to satisfactory results and, as a consequence, they give support to the idea that general equilibrium does not really provide a convincing explanation for economic fluctuations.

As we have seen, in one case (Grandmont's model) the criticism of "ad hoc" assumptions applies with regard to the specification of the expectation function; in the other case (Long-Plosser's model), the possibility of deviations from the steady state arises only

because of the ("ad hoc") assumption of random shocks to production possibilities. The latter shocks, together with optimal level of inputs, determine the amounts of commodities that are produced in the model economy. However, by assumption, there is no technical change and hence it is not too difficult to understand the "ad hoc" character of the introduction of these random shocks in the production functions.

A final remark seems necessary.

Although in Chapter 1.1 we have expressed the purpose of weighing the relative advantages and disadvantages of the equilibrium approach to business cycle theory as compared with those of the macro-behavioural approach, this problem has not been explicitly touched upon in the analysis we have developed above. This is because the conclusion that the equilibrium framework is not the proper framework for analysing business cycle phenomena cannot be drawn from a discussion of these advantages and disadvantages because the points of view and the objectives are far too different in the two approaches. For example, although the temptation is strong, it is of little interest if equilibrium models are criticized because of their reliance on exogenous shocks. Indeed, given the restrictive ("ad hoc") definition of business cycle¹, according to which the main problem in business cycle theory is to discover general rules about co-movements among different

1. See Vercelli, 1984.

aggregative time-series, the reason why these co-movements start is quite irrelevant.

Apart from the problem of the political implications of equilibrium models, with which it is not too difficult to be in disagreement, however, the main reason why we have drawn the above conclusion lies elsewhere: as we have seen, the assumption of market clearing and rational expectations leave little room for explaining fluctuations in economic activity. The only possibility that still remains open is to make "ad hoc" assumptions about the specification of the information set or, alternatively, and this amounts to the same, about the specification of the expectation function.

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CHAPTER 3: The Dynamic Version of the "Fixed Price" Approach

3.1. Introduction

To round off the picture of the most recent approaches to the analysis of macroeconomic fluctuations, we consider in this chapter what we have referred to in Chapter 1.1 as the "fixed price" approach. Given the purpose of our research, however, we need to discuss some general features of this kind of approach in order to explain why we are going to consider only certain models.

Firstly, given that our purpose is to compare alternative approaches to the explanation of macroeconomic fluctuations, we are only interested in formulations of the model which are dynamic. Thus, for example, we are not interested in Malinvaud's model (1977) but rather in its dynamic generalizations¹. With regard

1. See, for example, Blad, 1981; Blad-Zeeman, 1982; Honkapohja-Ito, 1980, 1983.

to the dynamic generalizations we shall analyse, the expression "fixed price" models is inappropriate because they are models in which, in order to describe the long term evolution of the economy, not only the adjustment in quantities is considered but also an adjustment in prices and wages. Our purpose in maintaining the expression "fixed price" models is to underline their common ground with Malinvaud's (1977) "fixed price" static model¹. In particular, to keep this in mind may be useful in evaluating the relevance that results obtained in such models, where dynamic considerations are introduced in a static framework, can have.

Secondly, although we are going to consider them as models which offer one of the possible explanations of macroeconomic fluctuations, it is important to keep in mind that there are problems in doing this. Indeed, for example, the demand and supply functions that are considered in these models are the individual's demand and supply functions. Then, aggregation problems are avoided by assuming that all individuals are described by the same

1. In other words, we see the models we analyse in this chapter as responding to Malinvaud's challenge in his 1977 book (p.34) where he writes:

"... I shall not speak of the search for a formal dynamic process that would correctly represent actual adjustments, a process in which prices would move slowly and quantities would in the meantime have to be made mutually consistent. Such a search is certainly challenging to mathematical economists but would lead us too far out of our subject"

(my emphasis).

characteristics¹.

In short, it seems unsatisfactory to refer to these kinds of models as "Macro-Dynamic" models. After all, they are nothing but models in which macrodynamic considerations are introduced in a (disequilibrium) static and individual choice-theoretic framework².

In our opinion, however, it is interesting to consider these models. In particular, the contributions to disequilibrium theory given by Blad (1981) and Blad and Zeeman (1982) seem to be the most useful for our purposes. Among other considerations, the main reason for this is that the analytical tools used in these papers are of undoubted mathematical interest and call to mind tools used to represent persistent fluctuations by authors belonging to the macro-behavioural approach. Thus, a direct appreciation and

1. See, for example, Blad, 1981, p.125.

2. There cannot be any doubt about this interpretation of models of the non-Walrasian type. Drazen, in his review of macroeconomic disequilibrium theory, expresses the following opinion which, I think, is certainly shared by all theorists belonging to this approach:

"... Explanations of macroeconomic phenomena will be complete only when such explanations are consistent with microeconomic choice theoretic behavior and can be phrased in the language of general equilibrium theory"

(Drazen, 1980, p.293).

See also Fitoussi, 1983, p.1, where the same passage of Drazen's article is quoted.

comparison of the role played by "ad hoc" assumptions in the macro-behavioural approach as opposite to the disequilibrium approach seems possible.

To understand the role of these recent (dynamic) contributions to the "fixed price" approach, it is useful to begin our analysis with a description of a rationed equilibrium economy in its static formulation as given by Malinvaud (1977). To ensure continuity with the rest of the chapter, however, we begin in the next section employing the notation introduced by Blad in his 1981 paper.

3.2. Disequilibrium Statics: Malinvaud's Model

In his 1977 book, Malinvaud's main purpose¹ is to study the problem of unemployment in a general equilibrium framework concerning an economy with rationing. In doing this, his attention is focused on the short-run for which prices are assumed to be fixed whereas quantities are assumed to adjust in order to make the actions of economic agents compatible².

1. See Malinvaud, 1977, p.4.

2. Thus, the framework chosen by Malinvaud is a general equilibrium framework in the sense that quantities traded in all markets are simultaneously determined but, given the short-run price rigidity, not in the Walrasian sense that demand equal supply. The reasons behind his assumption of fixed price and quantity adjustment are given by Malinvaud in 1977, pp.10-12.

Given the focus on short-run, the equipment of the representative firm is taken as given. Moreover, the only good that is produced in the economy is assumed to be perishable. In particular, the last assumption does not leave doubts about the static character of the model which, indeed, is explicitly "claimed" by Malinvaud himself in the following passage:

"... (t)he good should be considered as non-storable, which means that we shall neglect complications resulting from involuntary or voluntary stock accumulation (in any case, these complications could not be dealt with correctly in a static model)."

(Malinvaud, 1977, p.38; my emphasis)

In such a framework, studying the interaction between the utility maximization problem of the representative consumer and the problem solved by the representative firm, and parametrizing the solutions in terms of the price of the good (p) and the wage rate (w), Malinvaud obtains the well known graph in the (p, w) plane which is shown in Figure 3.1. As we see¹, the (p, w) plane results to be partitioned in three regions: C is the Classical Unemployment

1. For a more detailed description of the construction of Figure 3.1, see Malinvaud, 1977, p.78-87 and Blad, 1981, p.126. For example, in Malinvaud, 1977, pp.78-83, the determination of the domain in which classical unemployment prevails is extensively explained by the author who shows how the boundary AWB is determined. The results for the other two domains are, however, merely stated by him. It is important, as we shall see later, to keep in mind that the case in which the representative producer is rationed in both markets prevails on the boundary WC which separates the Keynesian Unemployment region from the Repressed Inflation region. See Blad-Zeeman, 1982, p.166.

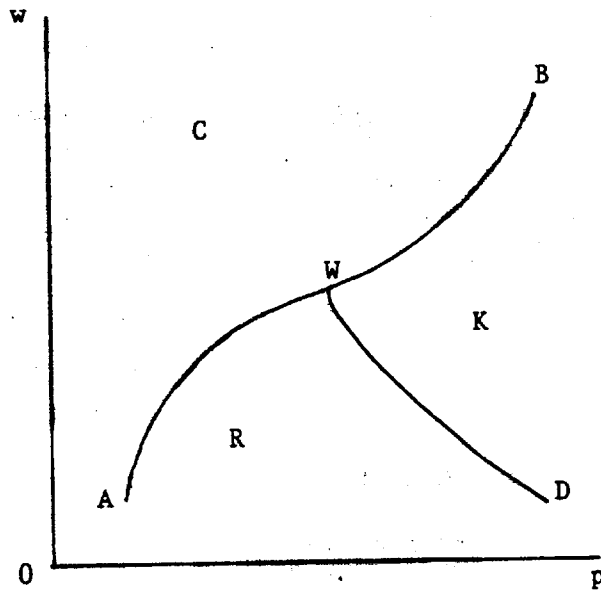


Fig.3.1

region; R, the Repressed Inflation region; K, the Keynesian Unemployment region; and W, where there is no rationing at all, is the Walrasian equilibrium.

Using Blad's formulation¹ rather than Malinvaud's original one, each state of the rationed equilibrium economy can be represented by a point in the product space $C \times X$, where $X = X(x, y, z)$ is the quantity space and $C = C(p, w)$, the parameter space. In particular, to distinguish the three regions in which the (p, w) space is partitioned by Malinvaud, we need a three dimensional quantity space where the x-coordinate measures the excess demand for the good; the y-coordinate, the extent to which notional output exceeds actual output; and the z-coordinate, the excess supply of labour.

In the context of a rationed equilibrium economy, three concepts of demand and supply are relevant.

For the representative consumer, the notional demand for goods and supply of labour are the demand $D(p, w)$ and supply $l(p, w)$ which prevail when there is no rationing at all. When, on the other hand, there is rationing, the consumer expresses his wishes taking into account this rationing. Thus, if D^a and l^a are his actual demand and supply, his effective demand for the good and supply of labour are $\bar{D}(p, w, l^a)$ and $\bar{l}(p, w, D^a)$ respectively. Analogously, for the representative producer, we can write $S(p, w)$, S^a , $\bar{S}(p, w, v^a)$, and

1. See Blad, 1981.

$v(p,w)$, v^a , $\bar{v}(p,w,S^a)$ for the three concepts of supply of goods and demand for labour respectively¹.

Using this notation, and employing the above mentioned concepts of demand and supply, we can easily distinguish the three regions considered by Malinvaud.

For equilibria in the K region, where the consumer is rationed on the labour market and unrationed on the goods market whereas the producer is rationed on the goods market and unrationed on the labour market, we have:

$$D^a = \bar{D}(p,w,l^a) < D(p,w),$$

$$l^a < \bar{l}(p,w,D^a) = l(p,w),$$

and

$$S^a < \bar{S}(p,w,D^a) = S(p,w),$$

$$v^a = \bar{v}(p,w,S^a) < v(p,w).$$

Thus, in terms of the coordinates of the quantity space C , we can write:

1. This means that the notional demand/supply express the utility/profits maximizing quantities; the actual demand/supply express the expected realized quantities; and the effective demand/supply express the quantities, when the rationing on the other market has been taken into account. See Blad-Zeeman, 1982, p.180.

$$x = \bar{D} - S^a = D^a - S^a = 0,$$

$$y = S - S^a > 0,$$

$$z = \bar{l} - v^a > 0.$$

Thus, an equilibrium in the K region is represented by a point in the product space $C \times X$ such that $e_k = (p, w, 0, y, z) \in C \times X$ with $y > 0, z > 0$.

Following an analogous procedure, for an equilibrium in the C region, where the consumer is rationed on both markets and the producer is not rationed at all, we find that such an equilibrium is represented by a point $e_c = (p, w, x, 0, z) \in C \times X$ with $x > 0, z > 0$.

Finally, an equilibrium in the R region, where the consumer is rationed on the goods market and unrationed on the labour market whereas the producer is rationed on the labour market and unrationed on the goods market, is $e_r = (p, w, x, y, 0) \in C \times X$ with $x > 0, y > 0$.

This characterization of the equilibrium situations in a rationed economy given by Blad does not differ crucially from that given by Malinvaud except for the formalization in terms of points of the product space $C \times X$.

To notice this at this point is important because it allows us to make a preliminary statement about the type of models we will analyse in the next Section.

Indeed, this means that in Blad's (and Blad-Zeeman's) as in Malinvaud's model there are no inventories. As a consequence,

firstly, as underlined by Honkapohja and Ito¹, in the model there are imbalances between supply and demand but these imbalances cannot reveal themselves in the behaviour of inventories. Given the important role that is played by changes in inventories in macroeconomic fluctuations, this characteristic seems to reduce the importance of the exercise undertaken in the models we will analyse in the next Section. Secondly, leaving out the role of changes in inventories, the (p,w) plane may be partitioned into three regions rather than four because in this case the region in which producers are rationed on both markets is not especially interesting and can thus be neglected². However, the situation in which producers are rationed on both markets still prevails on the boundary between the K region and the R region and this is the reason why, as we shall see, problems arise in analysing crossing of the boundary between the two regions. Thirdly, as underlined by Malinvaud himself, changes in inventories could provide an important link between two successive periods so that it is puzzling that these changes have been disregarded in models the purpose of which is to dynamize

1. See Honkapohja-Ito, 1980, p.184.

2. Indeed, if producers are rationed on the goods market and the good is perishable, there is no reason why they should demand more labour than they can recruit.

Malinvaud's static framework¹.

In the latter framework, attention is focused on the short-run point of view, where prices and wages are assumed to be constant and equilibrium is attained only by (instantaneous) adjustments in the quantities².

On the contrary, the contributions we shall analyse in the next Section, in order to describe the long-run evolution of the economy, also introduce into the picture (long term) adjustments in prices and wages and consider adjustments in quantities that are "fast" rather than instantaneous.

3.3. "Fast" and "Slow" Dynamics: Blad's Paper

1. In Malinvaud, 1977, p.86, we read:

"... stock accumulation and decumulation does play an important role in business fluctuations and has to be taken into account in any dynamic analysis".

To understand the role that inventories play in a dynamic model of a rationed economy, it is also useful to notice that in Honkapohja-Ito's model, where imbalances between supply and demand reveal themselves in changes in inventories, the following is true:

"... The trading of goods is in turn determined as the minimum of supply and demand while their difference determines the initial inventories of the following period. This serves as the main source of dynamics in the model"

(Honkapohja-Ito, 1980, p.186; my emphasis).

2. See Malinvaud, 1977, pp.10-12.

According to Blad's formulation, price and wage rates react "slowly" to excess demand and supply whereas quantities react at a "faster" rate; i.e., price and wage rates react with a "low" speed of adjustment rather than not reacting at all as in Malinvaud's model, and quantities with a "high" speed of adjustment rather than infinite¹.

Given that the excess demand for goods (labour) and the excess supply of goods (labour) are different in the three regions, the "slow" dynamics in prices and wages must be specified separately for each of the three regions.

Making use of Blad's notation which we have used in the previous section, we can write the equations determining the "slow" dynamics in prices and wages in the following way²:

$$\begin{aligned}
 (1) \quad (p, w) \in K: \dot{p} &= K_p(\bar{D} - \bar{S}) = K_p(\bar{D} - S) = \\
 &= K_p(x - y) = -K_p y < 0, \\
 \dot{w} &= K_w(\bar{v} - \bar{l}) = K_w(v^a - \bar{l}) =
 \end{aligned}$$

1. Former articles in which a long-term adjustment process in prices and wages is introduced in a rationed economy model are Benassy's 1977 article and Blad-Kirman's 1978 article ("The Long-Run Evolution of a Rationed Equilibrium Model", Economic Research Paper n.128, University of Warwick). In these articles, however, the adjustment in quantities is assumed to be instantaneous and, as a consequence, the evolution of the economy is completely determined by changes in prices and wages. Blad's distinction between "fast" and "slow" dynamics can be seen as a way to overcome this difficulty.

2. The adjustment process in prices and wages is presented by Blad (1981, p.129) without discussion while the notion of "fast" and "slow" dynamics is extensively discussed by Blad (1981, pp.123-124).

$$= -K_w z < 0;$$

$$\begin{aligned} (2) \quad (p, w) \in C: \dot{p} &= C_p(\bar{D} - \bar{S}) = C_p(\bar{D} - S^a) = \\ &= C_p x > 0, \\ \dot{w} &= C_w(\bar{v} - \bar{l}) = C_w(v^a - \bar{l}) = \\ &= -C_w z < 0; \end{aligned}$$

$$\begin{aligned} (3) \quad (p, w) \in R: \dot{p} &= R_p(\bar{D} - \bar{S}) = R_p(\bar{D} - S^a) = \\ &= R_p x > 0, \\ \dot{w} &= R_w(\bar{v} - \bar{l}) = R_w(v - v^a - z) = \\ &= R_w(v - v^a) > 0; \end{aligned}$$

where the coefficients K_p , K_w , C_p , C_w , R_p , and R_w are all positive.

As underlined by Blad, as long as the price-wage rate combination stays within one region, the complete picture of the long-term evolution of the economy can very easily be obtained by analysing the interaction of this "slow" dynamics in prices and wages with the "fast" dynamics in quantities¹.

Before considering the specification chosen by Blad for the "fast" dynamics, however, it is necessary to understand that problems arise in the analysis of the case in which (p, w) moves from one equilibrium region to another. In fact, the analysis developed by Blad in this paper covers only the cases of crossing

1. See Blad, 1981, p.130, where the long-term evolution of the economy in this case is described.

of the boundaries between the C and the K regions and between the C and the R regions. The reason for this is evident if we consider the picture of the dynamics of the system in the (p,w) plane which is shown in Figure 3.2. As we see, given the formulation chosen by Blad for the dynamics of prices and wages, if we assume, as Blad does, that $K_w = C_w$ and $C_p = R_p$, both the vector field and its direction are continuous on the boundary AWB. On the contrary, both the vector field and its direction are discontinuous on the boundary WD¹.

To notice the existence of this problem awakens serious doubts about the usefulness of the exercise which is undertaken in this kind of model.

As a result of this discontinuity, since the analysis based on the consideration of "fast" versus "slow" dynamics does not cover the case of crossing of the boundary between the R and the K regions, this latter case is neglected by Blad. In a later paper written along very different lines², he (and Zeeman) offers an analysis of this case.

1. This property of the boundary between the R and the K regions seems to be the consequence of the fact that, given Malinvaud's assumptions, this boundary is the only part of the (p,w) plane on which the situation of producers rationed on both markets prevails.

2. See Blad-Zeeman, 1982. This solution is based on the consideration of the possibility of "inertia" in the process of decision making by firms and consumers as represented by the presence of time-lag in agents' response to changes in prices and wages.

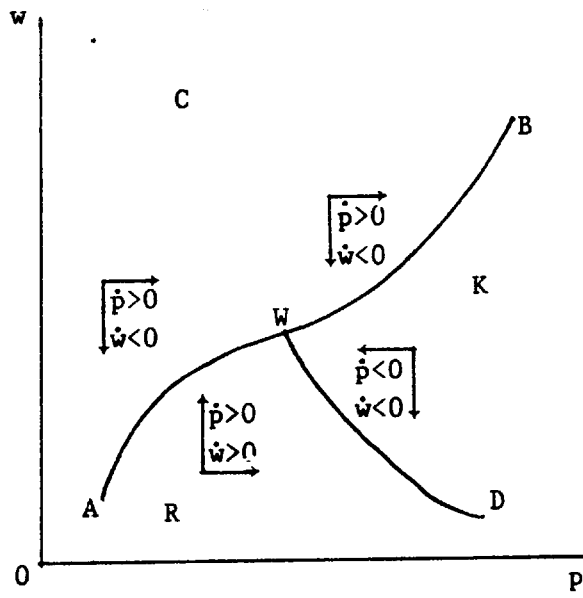


Fig.3.2

This is nothing but the confirmation of a certain "uneasy feeling"¹ that in this kind of model, where the purpose is to introduce a smooth dynamic formulation of the long-term evolution in a system which is inherently static, it is necessary to have recourse to expedients, in the form of "ad hoc" assumptions, in order to ensure that the analysis is applicable to the whole (p,w) plane.

This uneasy feeling appears to be reinforced if we turn our attention to the specification of the "fast" adjustment in quantities for the simplified case analysed by Blad in Section 5 of his 1981 paper².

In this simplified example, a specific trajectory, which crosses the boundary between the C and the K regions, is considered with a parameter γ representing the (p,w) value along it; to simplify, since in both regions consumers are rationed on the labour market it is assumed by Blad that z , the excess of supply of labour, is fixed along the trajectory and equal to one.

In defining the "fast" adjustment in quantities, Blad's purpose is that of satisfying the basic principle according to which quantities adjust smoothly to changes in the parameter rather than

1. See Fitoussi-Velupillai, 1984, pp.4-5.

2. See Blad, 1981, Section 5, "A First Approach to a Long-Term Evolution", pp.130-139. A "more general" situation is then analysed by Blad in the Appendix, pp.139-144.

by means of instantaneous jumps¹.

Given this purpose, in order to describe the "fast" adjustment in quantities, a potential function is used which, although it makes the economic interpretation "less obvious", is the "... simplest dynamics having the required form" (Blad, 1981, p.133)²:

$$V(\xi, \eta, z) = k\left\{\left(\frac{\xi^3}{3} - \gamma^2 \xi\right) + \left(\frac{\eta^2}{2} + \gamma \eta\right) + \frac{(z-1)^2}{2}\right\},$$

where $k > 0$, and ξ and η are new coordinates such that:

$$\xi = x + y,$$

and

$$\eta = x - y.$$

1. See Blad, 1981, p.132. As we will see, this leads quite naturally to a characterization of the long-term evolution in terms of "exchanges of stability" around the Walrasian equilibrium.

2. In introducing this assumption, Blad himself is forced to admit that

"... Unfortunately there exists no general accepted economic theory for such fast quantity adjustments to use as a guideline to the formulation. Accordingly, we shall state mathematically convenient equations, which give the required properties and then we shall discuss the economic content below"

(Blad, 1981, p.133).

Accordingly, the adjustment process in quantities is assumed to be represented by the gradient of the potential:

$$\dot{\xi} = -k(\xi^2 - \gamma^2),$$

$$\dot{\eta} = -k(\eta + \gamma),$$

$$\dot{z} = -k(z - 1),$$

where it is assumed that $\gamma < 0$ in the C region and $\gamma > 0$ in the K region.

With this formulation, which seems to leave little room for an economic interpretation¹, it is not too difficult to show that the "fast" dynamics in quantities has two points of equilibrium: for $\gamma < 0$, there is one stable C equilibrium and one unstable K equilibrium whereas for $\gamma > 0$, the C equilibrium is unstable and the K equilibrium is stable². Thus, for this simple example, the description of the long-term evolution of the rationed economy, including the case of crossing of the boundary between the C and the K regions and between the C and the R regions, is complete. Indeed, the result of the contemporaneous working of the "slow" and "fast" dynamics, as long as (p, w) stays within one region, is well described by Blad in the following passage:

1. An analogous opinion is expressed by Gabisch and Lorenz (1987, p.201). In their opinion, this formulation lacks any kind of economic meaning.

2. For the proof of this statement, see Blad, 1981, pp.4-5.

"given the initial value (p_0, w_0) , a fast adjustment in the quantities will take place from the initial value (x_0, y_0, z_0) towards the corresponding equilibrium point on the equilibrium quadrant. During this fast adjustment the (p, w) value change very slowly as described by the slow process above. These slow changes initiate further adjustments in the quantities and so on."

(Blad, 1981, p.130).

When the parameter crosses one or the other of the two boundaries mentioned above, it is necessary to guarantee that the fast adjustment in the quantities continues to depend smoothly on the parameter. For this reason, Blad considers the case in which there is an "exchange of stability" such that, although the old equilibrium continues to exist, it changes from being stable to being unstable. At the same time, the new equilibrium in the other region changes from being unstable to being stable so that the evolution of the economy can continue along this new equilibrium¹.

Without going into the finer details of Blad's model, it is already possible to draw some conclusions about the problem of the introduction of considerations of long term evolution into a rationed economy disequilibrium model².

1. See Blad-Zeeman, 1982, pp.130-136.

2. The analysis of Blad's model we have performed is incomplete. Indeed, we have left out (1) the last part of the example developed by Blad in which the potential is modified by introducing an "hidden" parameter in order to obtain a "universal unfolding" of the critical point and (2) the general case analysed by Blad in the Appendix (pp.139-144). However, we think, we have already the possibility of drawing the conclusions in which we are interested.

As we have seen, the description of regime-switching by means of the consideration of "fast" and "slow" adjustments in quantities and in prices, because of the definition chosen for the long-term adjustment process in (p,w) , can only be applied to the crossing of the boundary between the C and the K regions and between the C and the R regions. For this reason, Blad, in his paper, limits his attention to these two cases. Moreover, even for these two cases in which the analysis applies, he introduces the fast dynamics in quantities in several "ad hoc" ways which leave little room for an economic interpretation.

The case which is left out of consideration, namely the behaviour around the Repressed inflation-Keynesian unemployment boundary is analysed by Blad (and Zeeman) in another paper which makes use of other very different "ad hoc" assumptions¹. In short, the problem in this paper is to "refine" the model so that the vector field, which has a discontinuity on the boundary between the R and the K regions, produces a smooth flow.

The problem is tackled by Blad and Zeeman noting that, in Blad's 1981 model as in general in all "standard" models of a rationed economy, the production sector is introduced by an aggregate production function, the latter being implicitly contained in the

1. See Blad-Zeeman, 1982. A different analysis of the case of a crossing of the boundary between the R and the K regions is performed by Honkapohja and Ito in a recent paper (1983) making use of the concept of Filippov solution. However, this paper being more concerned with problems of stability of the Walrasian equilibrium, has not been considered here.

equations describing the slow dynamics in prices and wages. This is now seen by Blad and Zeeman as an unsatisfactory idealization of real situations in which total production is the result of several smaller production units' decisions. For this reason, a time-lag is introduced by the two authors in both firms' and consumers' decision-making process on the basis of the assumption that the time-lag distribution defines a probability measure on $[0, \infty)$.

However, only very special examples are analysed by Blad and Zeeman who consider three possible time-lag distributions and two possible combinations of vector fields¹. Analysing different possible combinations of vector fields and time-lag distributions, some results are obtained as, for example, that in the case in which the timelag distribution is the bounded uniform distribution and the vector field is of "bang-bang" type, the economy will exhibit stable steady oscillations around the boundary or oscillations of decreasing amplitude and period if the timelag distribution is a negative exponential distribution and the vector field is bounded below.

In the authors' opinion, these results are important because they

1. The three types of timelag distribution that are considered are the (1) negative exponential distribution, representing the case in which the timelag is randomly distributed over the interval $[0, \infty)$; the (2) decreasing linear distribution; and the (3) bounded uniform distribution. The two examples of vector fields are the (a) bang-bang vector field, referring to the complete opposite behaviour in the two regions, and the (b) bounded below vector field. See Blad-Zeeman, 1982, pp.169-172.

"... clarify the validity of the often made claim that a basic feature of modern economies is the cycling motion between Repressed Inflation and Keynesian equilibria"

(Blad-Zeeman, 1982, p.181)¹.

However, although the exercise undertaken might seem to deserve attention because of the obvious mathematical interest of the paper, this would not seem to justify the attribution of such a deep significance to results obtained while analysing very special (possibly "ad hoc") cases. The problem is that there is a lack of general results and, again, the feeling is that this is the unavoidable consequence of the type of exercise which is undertaken in this kind of model, i.e., the introduction of dynamic considerations in a static framework².

There is, however, another aspect worth discussing, i.e., the relation between the solution given to the problem of regime-switching in rationed economy models and the solutions given to the problem of the representation of persistent macroeconomic fluctuations by authors in the macro-behavioural tradition.

1. See, for example, Malinvaud, 1977, pp.94-97.

2. It is also important to remember the arbitrariness of the partition of the (p,w) plane in a K region, a C region, and a R region. To understand this, it is useful to take account of the results obtained in Honkapohja-Ito's paper (1980) where the role of inventories in a rationed economy is considered. As a first result, the classification of short-term states of the economy is affected: the region in which producers are rationed on both markets gains importance while the C region disappears. See Honkapohja-Ito, 1980, pp.188-190.

Firstly, as we will see in the next Chapter, the solution given by Blad in his 1981 paper consists in formulating the problem of regime-switching as a (generalized) relaxation oscillations problem. To obtain this solution, the crucial assumption is that there exist different speeds ("fast" and "slow") of adjustment for quantities and prices. In particular, in an article by Georgescu-Roegen (1951) we find a description of the concept of relaxation which leaves no doubts about an interpretation of Blad's solution along these lines¹. With the formulation chosen by Blad, the basic feature of the discontinuity in regime is undoubtedly better focused than in nonlinear models within the macro-behavioural approach where the formulation makes use of van der Pol's equation².

1. Indeed, in Georgescu-Roegen's article (1951, p.301) we read that

"... a relaxation phenomenon takes place ... when the difference between the 'up' and 'down' swings is created by a certain discontinuity in the regime. Such a discontinuity will introduce a discontinuity in the speed of the movement (at least in size or in direction). Therefore, the movements related to each phase will be described by different functions"

(original emphasis).

2. See Goodwin, 1951. In Georgescu-Roegen's opinion, in a relaxation phenomenon the movement in each phase has to be described by different functions. However,

"... The aim of van der Pol's contribution was to approximate these two different functions by a single analytical

(Footnote continues on next page)

However, whereas the latter formulation is explicitly dynamic, at least in the sense of having the investment function as a main subject of analysis, the former is not and this seems to reduce the importance of the consideration of discontinuities.

Secondly, also the solution to the problem of crossing of the boundary between the R and the K regions given by Blad and Zeeman, based on the introduction of distributed timelags in the decisions making process of both producers and consumers, calls to mind other solutions to the problem of the persistence of macroeconomic fluctuations given by authors in the macro-behavioural tradition, above all, by Kalecki¹.

In Blad-Zeeman's model, time-lags are introduced in order to take account of the existence of inertia in the process of decision making by firms and consumers. For example, whereas the production

(Footnote continued from previous page)

function. This was achieved by considering the periodic solutions of the differential equations:

$$(d^2y/dt^2) + \epsilon(y^2 - 1)(dy/dt) + y = 0,$$

for large values of ϵ . By this procedure the analytical difficulty was solved from the practical point of view. But this veiled the real meaning of relaxation, which is the discontinuity in the regime"

(Georgescu-Roegen, 1951, p.301; my emphasis).

1. See, for example, Kalecki, 1935. See also Goodwin, 1951, in particular the section "Expanded Model with Investment Lag", pp. 237-243.

function implicitly contained in the slow adjustment processes in prices and wages of Blad's 1981 paper is an aggregated production function, this is not seen as a satisfactory assumption by Blad and Zeeman. In their opinion, it is important to take account of the fact that, in the real situation, total production is the result of the decisions taken by several firms. Given that the process of obtaining and incorporating correct information is very complex and time consuming, the various firms react with different speeds to changes in the values of the external parameters.

Thus, this gives a rationale for Blad-Zeeman's assumption according to which:

"... while some firms will be able to make production decisions at time t based on the actual prices and wages exactly at time t , other firms will base their decisions at time t on the former values of the external parameters at time $(t-\tau)$ for $0 < \tau < \infty$ "

(Blad-Zeeman, 1982, p.168)

However, whereas in Kalecki's dynamic theory time-lags are introduced to describe the investment decisions process, in Blad-Zeeman the capital stock is taken as given and, as a consequence, there is no investment function. As with regard to the consideration of discontinuities, this fact seems to reduce the importance of the consideration of time-lags.

To conclude, although in "fixed price" models attention is concentrated on the basic problem of stagflation, and while the task of distinguishing among different types of unemployment is accomplished, it still seems impossible to get any valuable result

in a framework that is not macrodynamic from the very outset, i.e., that does not start explicitly from an analysis of the process of accumulation and all its interaction with the unemployment rate, the inflation rate, and the (functional) distribution of income.

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SECOND PART: THE MACRO-BEHAVIOURAL APPROACH



CHAPTER 4) Business Cycles Modelling and the Van der Pol Oscillator: An Introduction

4.1. The Van der Pol Oscillator: Limit Cycles and Relaxation Oscillations

Following out treatment of that aspect of Blad's model which we mentioned in the previous chapter, namely, the fact that the consideration of "fast" vs. "slow" dynamics is equivalent to the consideration of a (generalized) relaxation oscillation problem, we can now introduce the discussion of models within the macro-behavioural approach in a way which ensures continuity with the analysis we have above developed.

To understand the relations between the two types of formalization, we introduce, in this section, the van der Pol oscillator. In doing this, however, our purpose is to emphasize

only those features of this oscillator which are considered to be of interest for applications in economic dynamics¹.

The van der Pol equation is a nonlinear second order differential equation of the form:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \quad (1).$$

where $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$, and $\epsilon > 0$.

Alternatively, the equation can be written in the form analysed by Lord Rayleigh:

$$\ddot{y} + \epsilon\left(\frac{\dot{y}^3}{3} - y\right) + y = 0 \quad (2).$$

Although (1) and (2) are mathematically equivalent when $x = \dot{y}^2$, it is useful to consider both forms because they illuminate

1. Two basic references for this kind of equation are Le Corbeiller 1936, 1960. See also Hirsch-Smale, 1974, pp.210-215, where it is given an useful introduction to van der Pol's equation from an electrical circuit theory point of view.

2. See Le Corbeiller, 1960, p.387. Differentiating each term of the left-hand side of equation (2) with respect to time, we obtain:

$$\ddot{y} + \epsilon(\dot{y}^2\ddot{y} - \ddot{y}) + \dot{y} = \ddot{y} + \epsilon(\dot{y}^2 - 1)\ddot{y} + \dot{y} = 0,$$

from which, writing x for \dot{y} , we obtain equation (1). For the analysis of both equations and of their relation see Le Corbeiller, 1936.

different aspects of the problem of the existence of persistent, oscillatory solutions.

Considering, for example, the equation written in the van der Pol form, it is possible, by analogy with the solution of a linear second order differential equation, to identify easily the consequences of the nonlinearity of the equation. This nonlinearity is reflected in the coefficient of the term in \dot{x} which represents the dampening operating in the system. In particular, whereas for a linear equation of the type:

$$\ddot{x} + 2a\dot{x} + bx = 0,$$

where $b > a^2$, the cyclical solution either tends to zero, when $a > 0$, or is of ever increasing amplitude, when $a < 0$, for the solution of equation (1) this is not the case¹. Indeed, in (1) there is a "nonlinear dampening" which is positive for values of x between -1 and $+1$, and negative for values of x less than -1 or greater than $+1$. In other words, when $|x|$ is "small", the dampening is positive so that $|x|$ increases whereas when $|x|$ is "large" the dampening is negative so that $|x|$ decreases. Qualitatively, therefore, we may well expect that the result of these counteracting forces is a self-sustaining oscillatory movement in x the existence of which, however, must be rigorously proved.

1. See Levinson-Smith, 1942, p.382.

This can be done in an illuminating way if we take the equation written in the Lord Rayleigh form and apply Liénard's graphical construction¹.

To perform this construction, let us consider equation (2) which, writing x for \dot{y} , so that $\ddot{y} = \frac{dx}{dt} = x(\frac{dx}{dy})$, can be written as:

$$x(\frac{dx}{dy}) - \epsilon(x - \frac{x^3}{3}) + y = 0,$$

i.e.:

$$x(\frac{dx}{dy}) - H(x) + y = 0 \quad (3),$$

where $H(x) = \epsilon(x - \frac{x^3}{3})$.

Drawing in the (y,x) plane the curve:

$$y = H(x),$$

and following the procedure described in Figure 4.1², we can construct graphically the limit cycle which represents the dynamics of the system.

1. See Liénard, 1928. See also Le Corbeiller, 1936, pp.363-367; Levinson-Smith, 1942, Minorsky, 1962, pp.101-107; Huntley-Johnson, 1983, pp.100-103. Liénard's graphical construction applies only to the equation written in the Lord Rayleigh form. For an extension of the construction to a broader class of second order equations, see Le Corbeiller, 1960, Appendix III, p.397-398.

2. See Le Corbeiller, 1936, p.364..

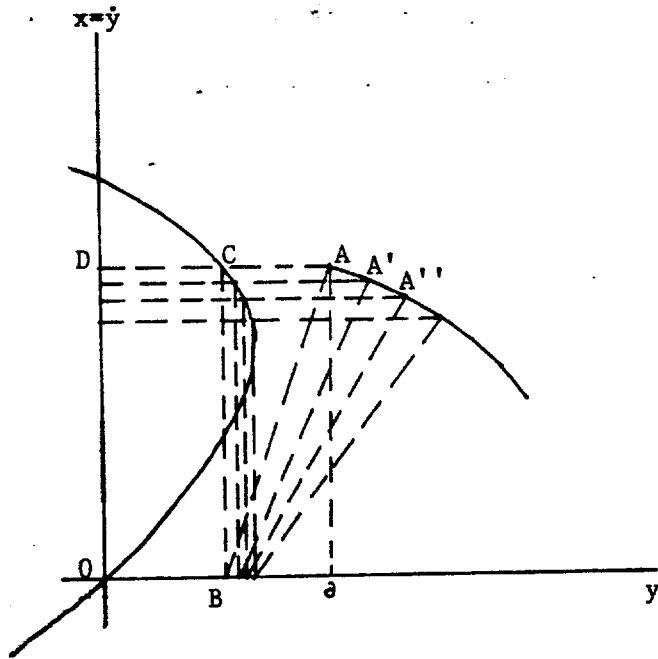


Fig.4.1

In Figure 4.1., AB is the normal to the curve (C) satisfying equation (3) and passing through any point A in the (y,x) plane.

Projecting BA on Oy we obtain:

$$\overline{Ba} = y - \varepsilon(x - \frac{x^3}{3}) = -x(\frac{dx}{dy}),$$

so that equation (3) can be written as:

$$\overline{Ba} = \overline{DA} - \overline{DC}.$$

Thus, we have the following procedure which allows one, for any point A in the plane, to obtain the tangent to the curve (C) passing through it: from A, draw ACD horizontal, CB perpendicular and join BA. The tangent to the curve (C) passing through A is perpendicular to BA.

In this way, repeating Liénard's construction for a neighbouring point A', A'' and so on, we obtain a family of curves, only one of which goes through any point of the plane, all curling inside and against the unique closed curve - the limit cycle - which is shown in Figure 4.2.

The radius vector \overline{OA} is such that:

$$\overline{OA}^2 = x^2 + y^2,$$

i.e., its second power is proportional to the sum of energies - T, kinetic, and U, potential - stored in the oscillation:

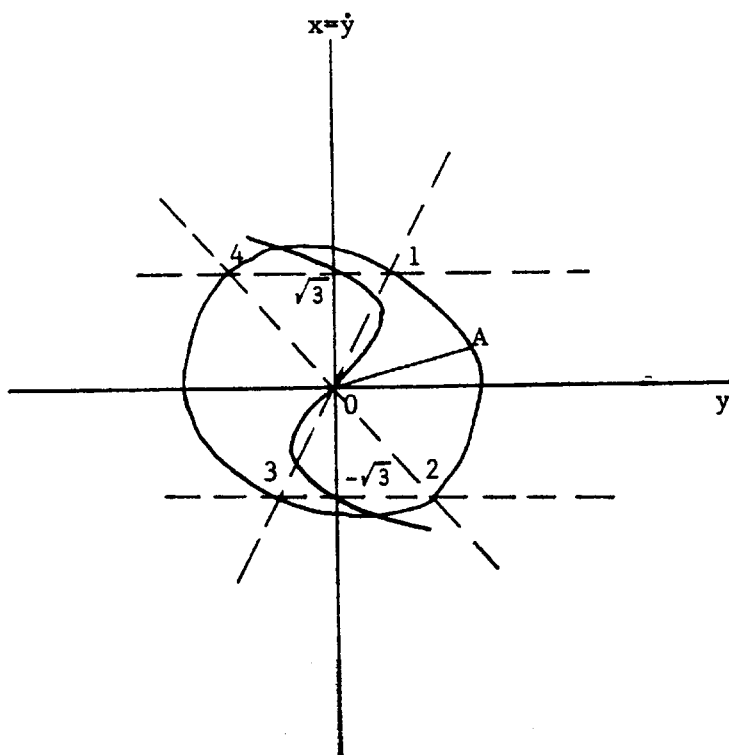


Fig.4.2

$$\frac{x^2}{2} + \frac{y^2}{2} = T + U = E.$$

As we see in Figure 4.2., from the point 1 to the point 2 (and from the point 3 to the point 4), the length of the radius vector \overline{OA} increases whereas from the point 2 to the point 3 (and from the point 4 to the point 1) it decreases. Thus, it is possible to write a (sufficient) condition for the existence of a limit cycle in terms of exchanges of energy between the oscillating system and the outside source. Given this interpretation, we have a limit cycle when, over a trajectory, the energies absorbed and dissipated cancel out, namely, when the following condition holds:

$$\oint \left(\frac{dE}{dt} \right) dt = \oint dE = 0,$$

where the curvilinear integral is taken along the trajectory¹.

Lord Rayleigh's equation can be written as:

$$\dot{y} = x,$$

$$\dot{x} = H(x) - y,$$

1. See Le Corbeiller, 1936, pp.371-372; Minorsky, 1962, pp.101-117; Levinson-Smith, 1942; Jordan-Smith, 1977, pp.320-344. In particular, the importance of the interpretation of Liénard's criterion in terms of energy exchanges is well underlined in Minorsky's book.

for which:

$$\begin{aligned}\frac{dE}{dt} &= x\dot{x} + y\dot{y} = \\ &= xH(x) = \\ &= \epsilon(\dot{y} - \frac{\dot{y}^3}{3})\dot{y},\end{aligned}$$

so that the (sufficient) condition for the existence of limit cycle becomes:

$$\oint dE = \epsilon \oint (\dot{y} - \frac{\dot{y}^3}{3}) dy = 0 \quad (4).$$

For example, the use of this criterion can easily be understood when we consider Lord Rayleigh's equation in the case in which ϵ is very small. In this case, we can expect the periodic solution to appear as a small distortion of one of the circular paths of the following linearized equation:

$$\ddot{y} + y = 0,$$

namely, we can expect the limit cycle to be close to one of the circles:

$$x^2 + y^2 = A,$$

where A is determined as approximate solution to condition (4)¹.

Indeed, for $\varepsilon = 0$, equation (2) becomes:

$$\ddot{y} + y = 0,$$

with solution:

$$y = A \cos t, \quad A > 0.$$

Thus, for $\varepsilon \approx 0$, we can approximately write:

$$\begin{aligned} dE &= \varepsilon \oint_0^{2\pi} \left(\dot{y} - \frac{\dot{y}^3}{3} \right) dy = \\ &= \varepsilon \int_0^{2\pi} \left\{ A \sin t - \left(\frac{A^3}{3} \right) \sin^3 t \right\} A \sin t \, dt = \\ &= \varepsilon \left[A^2 \int_0^{2\pi} \sin^2 t \, dt - \left(\frac{A^4}{3} \right) \int_0^{2\pi} \sin^4 t \, dt \right] = \\ &= \varepsilon A^2 \left[1 - \frac{A^2}{4} \right], \end{aligned}$$

which is equal to zero for $A = A_0 = 2$.

In the next chapter, we will utilize this criterion to obtain (sufficient) conditions for the existence of limit cycles in models within the macro-behavioural approach. To do this, we will employ a generalization of the criterion for an equation of (generalized)

1. See, for example, Jordan-Smith, 1977, pp.100-102; Arnold, 1979, pp.107-108; Minorsky, 1962, pp.116-117.

van der Pol's type¹:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

However, for the moment, we prefer to concentrate our attention on another feature of the van der Pol oscillator which allows us to link this oscillator to the formalization in terms of "fast" and "slow" dynamics we have considered in the previous chapter. Indeed, when the parameter ϵ is very "large", it is possible to show an important analogy between the two formalizations.

To do this, it is useful to consider the Liénard phase plane, where, to have the possibility of ascertaining the effect of the parameter ϵ , we have taken the ordinates equal to:

$$z = \frac{1}{\epsilon}[\dot{x} - H(x)].$$

With this notation, equation (1) is equivalent to the following system:

$$\dot{x} = z + H(x) \tag{1'}$$

$$\dot{z} = -\frac{x}{\epsilon} \tag{1''}$$

from which we obtain:

1. See Levinson-Smith, 1942.

$$\frac{dz}{dx} = - \frac{x}{\epsilon^2 [z - (\frac{x^3}{3} - x)]} \quad (5).$$

What we have obtained is a form of the van der Pol equation in which the effect of the parameter ϵ can easily be ascertained. For example, in the case in which $\epsilon \rightarrow +\infty$, equation (5) reduces to:

$$[z - (\frac{x^3}{3} - x)]dz = 0,$$

which, as is shown in Figure 4.3., is satisfied on the segments DA and BC of the curve - where $z = (\frac{x^3}{3} - x)$ - and on the segments AB and CD parallel to the x axis - where $dz = 0$.

When ϵ is very large, equations (1') and (1'') can be called the "fast" and the "slow" equation respectively because, in this case, the rate of change of x is much greater than that of z^1 . Moreover, from (1'), the integral curve consists of two branches which are traversed with different speeds: when the representative point follows the segments DA or BC - where $z = F(x)$ - the speed is

1. See, for example, Saunders, 1980, Ch.5, "Applications in Physics", pp.68-72. As stated by Saunders (p.70), however, it is important to stress that there is an important difference between elementary catastrophe theory and the formalization of (1) in terms of (1') and (1''): in the Blad model we have analysed in the previous Chapter, the basic concept is that of an exchange of stability that allows one to represent the whole dynamics of the system. Indeed, when the stable equilibrium point bifurcates it gives rise to an unstable equilibrium point with a stable one to either side. The van der Pol equation, on the other hand, undergoes a Hopf's bifurcation in which the unique unstable equilibrium is surrounded by a stable limit cycle.

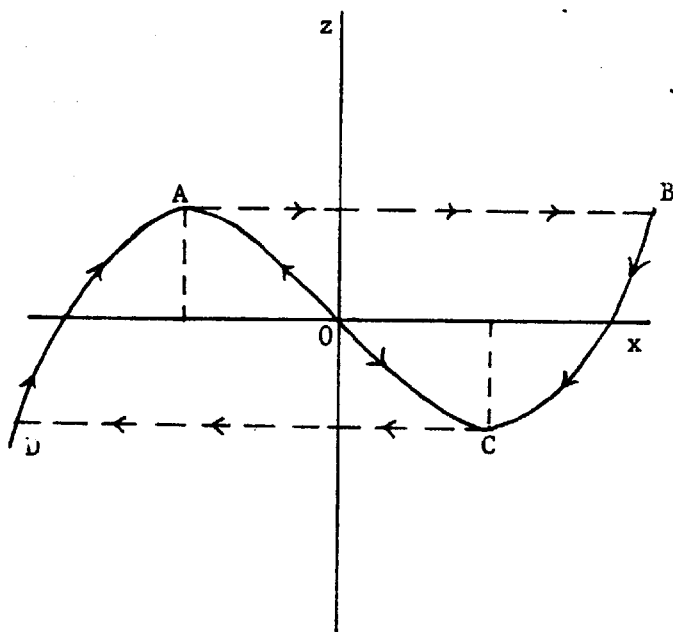


Fig.4.3

finite whereas when it follows the segments AB or DC - on which $z \neq F(x)$ - the speed is practically infinite¹. Thus, the basic idea behind the formalization of the dynamics of an economy in terms of van der Pol's oscillator or in terms of "fast" and "slow" dynamics is that there exist different "regimes" in each of which the dynamics of the system is governed by different equations².

Although in the previous chapter we have introduced this problem in connexion with very recent contributions to economic dynamics, this is an "old" idea in business cycle theory which is also worth analysing in relation with earlier contributions. In particular, it seems important to consider contributions of the late Twenties, early Thirties when the usefulness of the idea of relaxation oscillations to explain and describe economic processes was strongly advocated .

4.2. Hamburger's Contributions

Relaxation oscillations are usually associated with the name of Le Corbeiller and his article published in the first volume of *Econometrica* (1933) where the importance of a theory of

1. See Minorsky, 1962, p.111.

2. As we have already stressed at the end of the previous chapter, however, in the van der Pol's oscillator the two different functions are approximate by a single analytical function.

oscillations, independent of the field of application, was strongly advocated¹. However, we prefer to concentrate our attention on earlier contributions to the problem given by the Dutch economist L. Hamburger. Although these contributions are usually neglected, they present interesting aspects which make them worth analysing.

As early as 1928, in a discussion following a paper read by van der Pol and J. van der Mark at the 1928 Meeting of the Batavian Society of Logical Empirical Philosophy, Hamburger began to suggest the idea that the sequence of economic crises may belong to this category of oscillations. Although it took him some time to fully work out his idea, he finally achieved this as testified by a paper published in "De Economist" in 1930 and, in 1931, in French, in "Supplément aux Indices du Mouvements des Affaires"², where explicit recognition is given to the importance which mathematical tools developed in other fields could have for economic dynamics

1. In Le Corbeiller's opinion, given that the problems of fluctuations in economic activity is one of the most important and difficult to analyse,

"... il ne sera sans doute pas de trop, pour approcher de sa solution, de la mise en commun de toutes les ressources de la théorie des oscillations et de la théorie économique. C'est pourquoi j'ai pensé pouvoir vous présenter un compte-rendu succinct d'une avance récente, que je crois important, de la théorie des oscillations: celle apportée au problème des systèmes autoentretenues par la découverte des oscillations de relaxation, due à un savant hollandais, le Dr. Balth. van der Pol"

(Le Corbeiller, 1933, pp.328-329).

2. This is stressed by Hamburger himself in a Note in the Second Volume of Econometrica (1934, p.112). For the French version of the article, see Hamburger, 1931.

modelling. Although Hamburger does not provide us with a model and his approach to the problem consists in giving examples which support his convictions, it is interesting to analyse the reasons adduced by him to explain the advantages of a formalization in terms of van der Pol's equation.

In Hamburger's opinion, there exist many important phenomena of life for which the kind of oscillation differs definitely from harmonic oscillations. For all these phenomena, the resulting dynamics cannot be represented by the solution of a linear equation of the type:

$$\ddot{y} + \alpha \dot{y} + y = 0 \quad (1),$$

where, for example for application in physics, y is the deviation from the position of equilibrium; $\alpha \geq 0$, a quantity related to the resistance to which the moving mass is subjected; $\dot{y} = dy/dt$; and $\ddot{y} = d^2y/dt^2$.

The solution of equation (1) consists of harmonic oscillations of decreasing, constant, or increasing amplitude depending on the sign of the parameter α . When $\alpha < 0$, we have oscillations the amplitude of which increases without limit and which lead to a destruction of the system rather than to a permanent, periodic movement. However, in Hamburger's opinion, although this case must be rejected when we use a linear formalization, it is potentially the most interesting one for a representation of persistent oscillations. For example, in physics, where the instability of a system can represent a source of serious danger, one must take

preventive measures. In electrical circuits, for example, every time that there is an imminent danger, one can couple the original system with a positive resistance the value of which increases with the amplitude of the oscillations. Alternatively, one can incorporate in the original system an inverse force the value of which increases with the amplitude of the oscillations.

In both cases, the resulting movement can be described by the solution of van der Pol's equation:

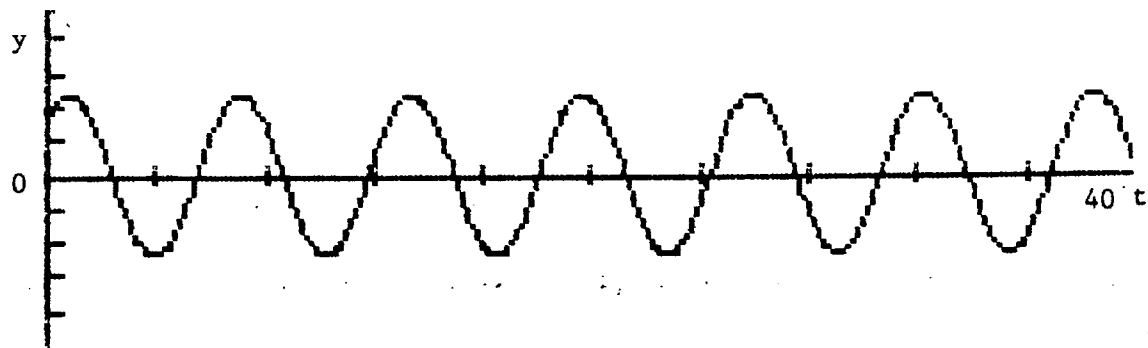
$$\ddot{y} - \alpha(1 - y^2)\dot{y} + y = 0 \quad (2),$$

where the resistance is (a nonlinear) function of the deviation from the position of equilibrium.

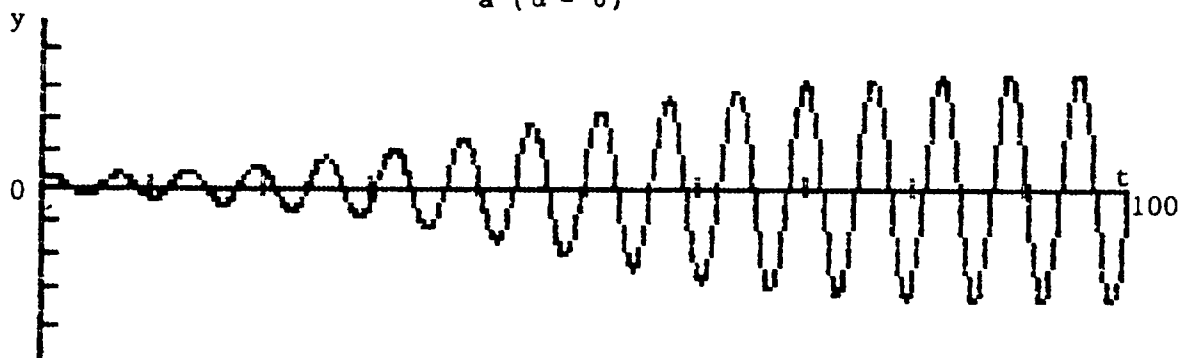
To understand what are the consequences for the dynamics of the system of such a nonlinear resistance we consider in Figure 4.4. a,b,c,d the representation of the resulting fluctuations for different values of the parameter α ¹.

The case shown in Figure 4.4.a, where $\alpha = 0$ and the oscillations are of constant amplitude, is the case in which (2) and (1) reduce to the same equation the solution of which represents a persistent, oscillatory movement of constant amplitude. However, whereas the solution of equation (1) represents explosive oscillations for any positive value of the parameter α , this is not the case when we

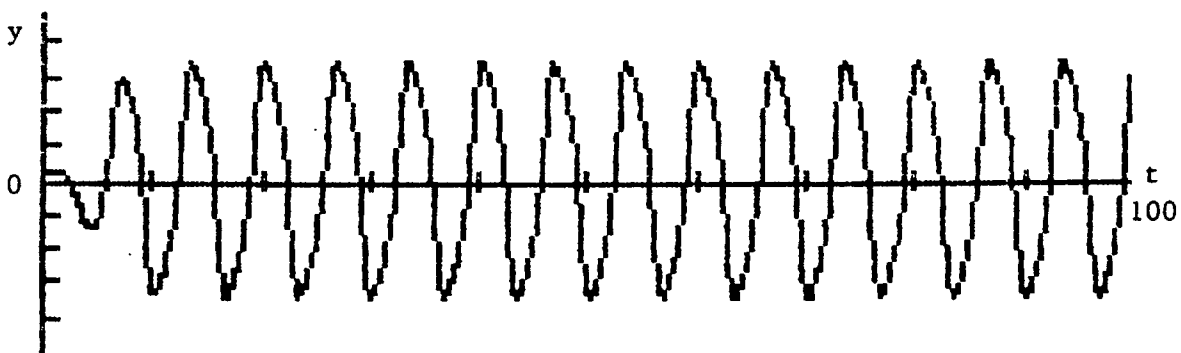
1. The graphs in Figure 4.4., as others which follow in the next chapter, have been drawn using PHASER, the program to simulate with differential and difference equations contained in Koçak, 1986.



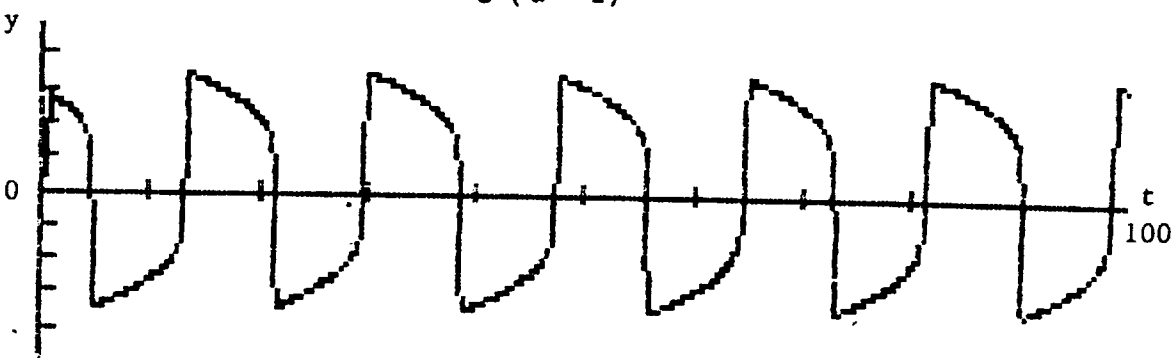
a ($\alpha = 0$)



b ($\alpha = 0.1$)



c ($\alpha = 1$)



d ($\alpha = 8$)

Fig.4.4

consider equation (2) as is shown in Figure 4.4.b - where $\alpha = 0.1$ - and Figure 4.4.c - where $\alpha = 1$. Although the solution consists of fluctuations which, at the beginning, are of increasing amplitude, they do not explode but eventually come to be of constant amplitude. Finally, when the parameter α is positive and very large, the solution of equation (2) shows fluctuations which quickly reach their constant amplitude as is shown in Figure 4.4.d. In this case, the source of energy is very strong and rapidly gives rise to important deviations from the position of equilibrium. However, given that the stronger the inverse force the larger is the deviation from equilibrium, there will be a violent recall sooner or later in the opposite direction. As a result, we have a periodic movement of "charge" and "décharge" which is very different from a sinusoidal movement and which, in Hamburger's opinion, has an important meaning for a "new method of analysis of economic crises"¹. To understand why this is so, it is enough to list some of the characteristics of this kind of oscillations which van der Pol has called relaxation oscillations. Firstly, their period of oscillation, or time of "charge" and "décharge", is given for a given system with a given resistance. As a consequence, there are drastic changes of the period every time that even only the resistance changes. Moreover, this also happens when exogenous and periodic forces are superimposed on the original system. Although these forces affect the periodicity of the system of relaxation,

1. See Hamburger, 1931, pp.7-8.

however, they do much less so with regard to the amplitude of the resulting oscillations. Secondly, relaxation oscillations arise every time that in the system there exists a source of energy which could lead to unstable positions but the action of which is periodically damped by an increasing resistance. Thirdly, the resulting oscillatory movement is very different from a sinusoidal movement in that there are sudden "jumps" whenever an unstable position is reached.

In Hamburger's opinion, these are all features which characterize fluctuations in economic activity. In particular, given the complexity of the economic system, it is quite remarkable how the original oscillatory movement maintains its amplitude even when exogenous forces are superimposed. Given this property, although some factors which influence the final result exhibit oscillations of sinusoidal type - and there is no doubt about this - it is enough to show that there are other factors which give rise to vigorous relaxations to be sure that the global motion of the economic system will show all basic characteristics of relaxation oscillations¹.

1. Apart from the influence of periodic factors such as the succession of seasons, there are other factors which influence sales and which present all characteristics of relaxation phenomena. For example, these are all factors which cause what we can call "artificial" or "conventional" seasons such as the one described by Hamburger in the following passage (pp.14-15):

To support his idea, Hamburger gives many examples of oscillations which can be interpreted in terms of relaxation oscillations. In one of these examples, he describes the cycle characterizing the behaviour in time of output as a sequence of phases of "charge" and "décharge"¹. In his opinion, the dynamics of total output must be described in terms of relaxation oscillations because the different phases of the process of production are inevitably characterized by different speeds at which changes take place². For example, in the phase of "construction", i.e., in the phases of recovery and expansion, there are physical factors which limit the attainable rate of

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"... une grande partie de la population faisait des économies pendant toute une année, afin de dépenser ("se décharger") lors d'une période de fête (foire). Il en est ainsi encore, ..., alors que le cadeaux, que l'on a l'habitude d'écharger aux fêtes, telles que Noel ou Saint Nicolas, déterminent des pointes dans la courbe des ventes. Après chaque maximum il y a une chute brusque et alors commence une nouvelle période d'une durée de presque une année, pendant laquelle on fait des économies".

(Hamburger, 1931, pp.14-15).

1. See Hamburger, L., 1931, pp.19-22.

2. For this reason, we need a nonlinear formalization. As stressed also by Goodwin in a contribution which we will analyse in the next chapter, one of the most important advantages of a nonlinear theory of the business cycle is that with such a theory:

"... we may make the depression as different from the boom as we wish"

(Goodwin, 1951, p.4)

growth. On the contrary, such physical factors do not operate to the same extent in the phase of depression with the consequence that there can be a sudden fall in the total supply of the economy.

The phase of "construction" can be represented by means of a logistic curve as the one shown in Figure 4.5¹ where it is not too difficult to account for the point of inflection. For example, as stressed by Hamburger, we can say that the slow-down of the rate of growth is inevitable given that, to exploit all possibilities open by the period of growth, people resort more and more to inferior instruments of production. This, however, cannot go on indefinitely. Sooner or later, following the slow down of the rate of growth, there is an inevitable crisis which is necessary in order to restore the relation between total supply and total demand. In this phase, the behaviour in time of output can be represented by the dotted line we have drawn in Figure 4.5. Once the relation between supply and demand has been restored by the "selection" operated during the crisis, a phase of growth can start once more with the consequence that the endogenous mechanism

1. See Hamburger, 1931, p.20. We have obtained the curve shown in Figure 4.5 as solution, equal to $y = k/(1 + Ce^{-at})$ - of a nonlinear differential equation of the type:

$$\dot{y} = ay(1 - \frac{1}{k}y).$$

See, for example, Davis, 1962, pp.96-98.

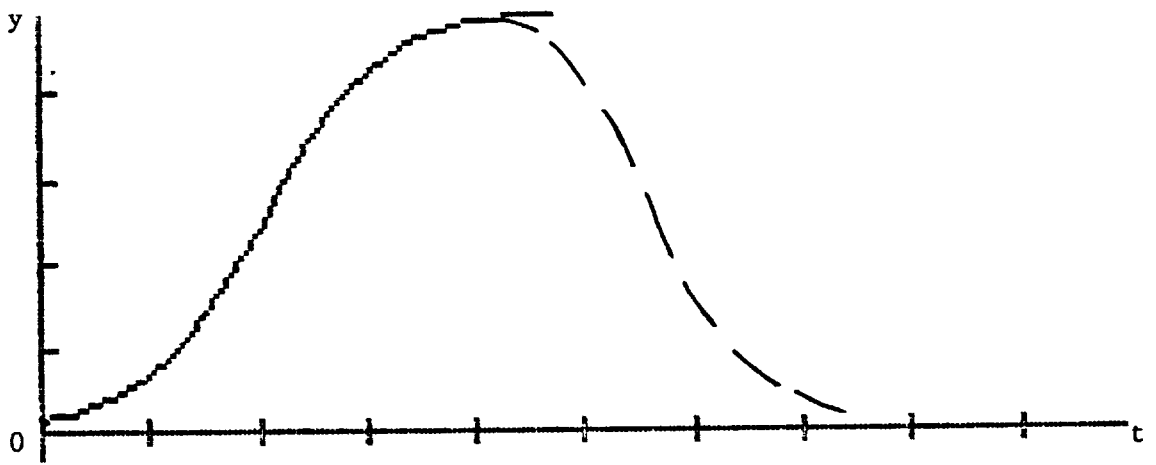


Fig.4.5

described above will repeat all over again¹ as is shown in Figure 4.6..

Thus, Hamburger's approach consists, firstly, in describing verbally the behaviour in time of output and, secondly, in drawing an ideal graph of such a dynamics in a way that the figure strongly resembles the solution of van der Pol's equation for the case in which $\alpha \longrightarrow +\infty$ ². In particular, in doing this, his intention is that of giving an endogenous explanation for the cyclical behaviour of the economy. Moreover, his explanation refers explicitly to the whole economy, i.e., it is an explanation from the macro-economic point of view. For both reasons, therefore, his approach is nearer in spirit to the macro-behavioural approach which we will analyse in the next chapter than it is to the most recent approaches which we have analysed in the previous ones.

This is true also with regard to the mathematical tools which are employed for the formalization of the models given that, in both models we will analyse in the next chapter, the dynamic

1. In Hamburger's opinion there is an evident analogy between this description of the fluctuations in total production of the economy and the evolution in time of the number of elements of two competing species. Indeed, in a period in which the number of prey increases, the number of predators increases too. However, although the rate of growth is very low at the beginning, it becomes the more and more rapid with the consequent destruction of prey. Thus, predators can find no longer enough nutriment and their species is decimated. As a consequence, it once more gives rise to a period in which the number of prey increases and so on. See Hamburger, 1931, pp.20-21,

2. Note that to obtain Figure 4.6. it is enough to smooth the corners of the path shown in Figure 4.4.d.

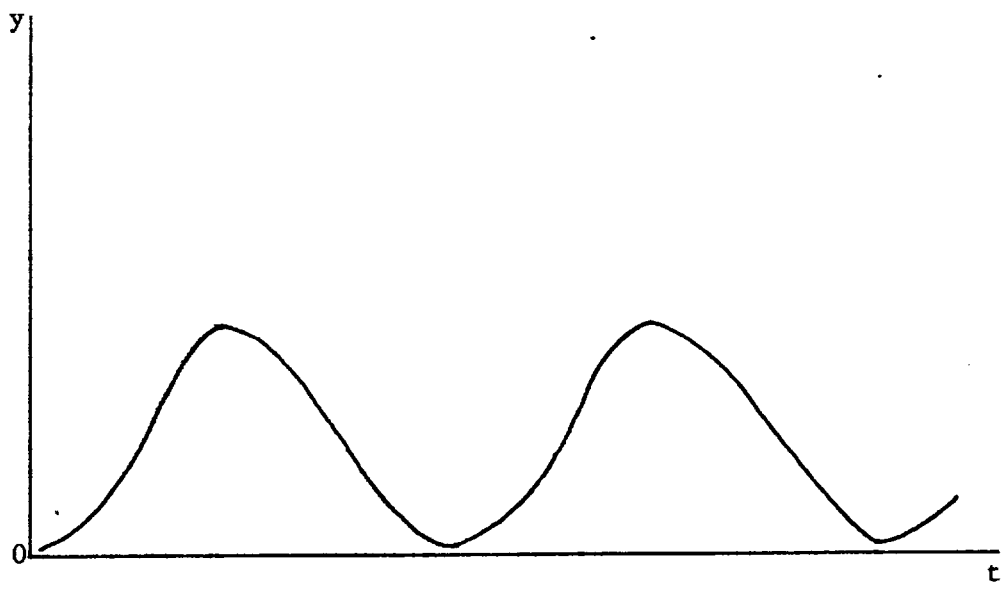


Fig.4.6

system can be reduced to an equation of van der Pol's type¹.

However, there is an important difference which is worth stressing. Given the approach chosen by Hamburger, and given that his contribution is a very early one, his description of economic fluctuations in terms of relaxation oscillations is a purely verbal one. The lack of formalization, in particular, makes itself felt with regard to the plausibility of the explanation of the lower turning points of the cycle, i.e., of the explanation of the reasons why the economy recovers from the slump and the cycle repeats itself again.

The approach implicit in the contributions we will analyse in the next chapter is by contrast quite different. In particular, as in Hamburger's contribution, so too in Goodwin's the usefulness of a formalization in terms of van der Pol's oscillator, and in general of nonlinear theory, is explicitly recognized². However,

1. An other important analogy can be found in the example given by Hamburger in terms of the dynamics of two competing species and the model developed in Goodwin, 1967.

2. However, Goodwin makes reference to Le Corbeiller's contribution rather than to that of Hamburger. In stressing the necessity of seeking in nonlinearities an explanation for the maintenance of oscillation, he says that:

"... (a)dvce to this effect, given by Professor Le Corbeiller in one of the earliest issues of this journal, has gone largely unheeded"

(Goodwin, 1951, p.2).

in Goodwin's paper, the idea is formalized right from the beginning and the approach shows an opposite orientation: the various equations are manipulated and simplified in such a way that the resulting final equation, describing the dynamics of the economy, is an equation of van der Pol's type to which all standard methods can be directly applied¹. Thus, some of the assumptions may well

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This reference to the name of Le Corbeiller, however, should not be any cause for surprise given the friendship between Goodwin and Le Corbeiller established at Harvard University and documented, for example, in the interview given by Goodwin to M. Palazzi (1982, pp.18-20). Goodwin's 1951 article owes much to the stimulus given by Le Corbeiller and this is clearly recognized by Goodwin himself in the following passage:

"... (m)y debt to Professor Le Corbeiller is very great, not only for the original stimulation to search for the essential nonlinearities but also for his patient insistence, in the face of the many difficulties which turned up, that this type of analysis must somehow be worked out"

(Goodwin, 1951, p.2f; original emphasis).

1. In very recent contributions, for example, in Goodwin 1986, a different orientation which, in a sense, is nearer in spirit to that of Hamburger, is shown. One of the examples given by Hamburger to support his idea that economic fluctuations belong to the class of relaxation oscillations refers to the process of innovation. In his opinion, the periodicity with which inventions are made is nothing but a phenomenon of "charge" and "décharge" in which the energy of a population, accumulated "dans les hommes supérieurs", is set in motion by means of vigorous pulses. Every time that an innovator is successful, his example is followed by many others. However, sooner or later, when the number of the enterprises is excessive and their coordination is imperfect, there is a sudden interruption of the growth. This can be seen as an interesting description of the phases of "charge" and "décharge" of a single

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be seen as "ad hoc" in the sense that they are assumptions which are made to reach a pre-determined result.

The next chapters are devoted to the analysis of this problem.

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wave. However, the main problem to be explained seems to be another one which is not even mentioned in Hamburger's paper, namely, to explain why the completion of a single wave leads to a situation such that the innovative process starts all over again. The contribution given by Goodwin in his 1986 paper goes in this direction given that his purpose, as explained by the author at the beginning of the article, is, first, to approach the problem of how single wave arises from a logistic, and, second, to approach the problem of how and why the completion of each wave leads to a situation which engenders a succeeding wave. See Goodwin, 1986, p.358 and 363.

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CHAPTER 5) The Macro-Behavioural Approach to Economic Fluctuations .

Analysis: Non-Linear Investment Functions and the Persistence of Economic Fluctuations

5.1. Introduction

To understand the importance which nonlinear theory can have in the formalization of a dynamic model of the economy, we consider in this chapter some contributions to what we have referred to in Chapter 1.1 as the macro-behavioural approach. In particular, we will concentrate our attention on contributions made by two authors belonging to this approach, namely, Goodwin and Kaldor¹. Although in doing this we leave aside contributions which are equally important, there are a number of arguments which seem to justify our choice.

1. In particular, we will consider the nonlinear models contained in Goodwin, 1951 and Kaldor, 1940.

Firstly, Goodwin's 1951 model, which was originally presented at the 1948 meeting of the Econometric Society in Cleveland and at the time summarized in *Econometrica*, 1949, has much in common with the model presented in Hicks's 1950 book. In particular, the two authors are both dissatisfied with linear models of the business cycle and both, as an alternative, propose models in which the nonlinear working of the acceleration principle is considered. The reason why we have decided to analyse the version of the model given by Goodwin is that this version, unlike that of Hicks, explicitly takes into account the nonlinear functioning of the acceleration principle in the formalization of the investment equation and thus the resulting dynamics is susceptible to an analysis along the lines we have described in the previous chapter.

Secondly, Kaldor's 1940 model is based on principles very similar to those on which the first version of Kalecki's business cycle theory is based as is stressed by Kaldor himself in the Appendix to his 1940 article¹. In particular, this is true with regard to the analysis of the factors determining investment behaviour. For example, in both models, it is assumed that, for a given level of the stock of capital, investment depends on the level rather than on the rate of change of output. In both cases, this assumption is the result of the rejection of the "general validity" of the acceleration principle which may be interpreted

1. For analysis of successive versions of Kalecki's business cycle theory, see Sordi, 1986.

along the lines suggested by Tinbergen¹. However, it is preferable to work with the version of the model given by Kaldor because his model has been at the center of discussion in the last years; as a consequence, we are now able to study and appreciate it along more modern lines².

Thirdly, given the recent contributions to the problem by Benassy, the decision to concentrate our attention on these two models will allow us to go back to the problem of the role played by ad hoc assumptions in the alternative approaches. In these contributions, Benassy raises the problem of the "ad hoc" character of the specification of the nonlinearities in Goodwin's and Kaldor's models³ and offers an alternative explanation of the persistence of fluctuations in economic activity. This alternative explanation is based on a model with "important novel elements" which has the purpose of showing how cycles can be generated using the "traditional shapes" for the various functions without having

1. See, for example, Tinbergen, 1938. Tinbergen can be seen to have played a crucial role in the development of the macro-behavioural approach to business cycle theory. Moreover, in saying this, we must also take into account his 1931 model which contains an analysis of a mixed differential-difference equation later on used by Kalecki and his 1944 article which is quoted by Goodwin as an example of work on nonlinear theories. See Kalecki, 1935; Goodwin, 1951, p.2f.

2. For example, different mathematical aspects of Kaldor's model are analysed in Ichimura, 1954; Chang-Smyth, 1971; Klein-Preston, 1969; Kosobud-O'Neil, 1972; Varian, 1979.

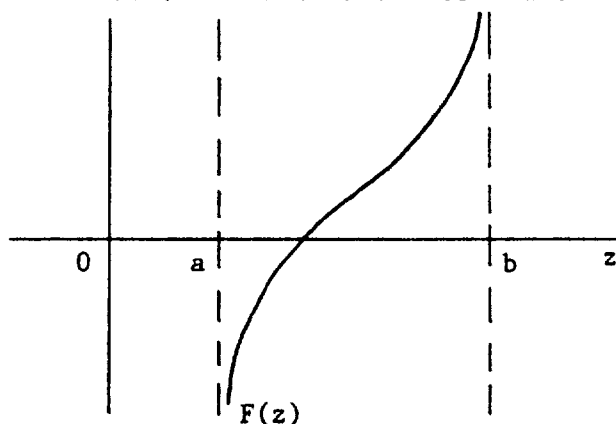
3. See Benassy, 1984, 1986.

"... to make fairly ad hoc assumptions such as sigmoid shapes for the investment function or the Phillips curve"

(Benassy, 1984, p.78; my emphasis).

In order to understand the relevance of Benassy's criticism¹, we consider in the next Section the 1951 model by Goodwin. Our purpose in doing this is to analyse the role which is played in this model by the choice of the shape of the nonlinear investment

1. Whereas the first criticism made by Benassy refers to Goodwin's and Kaldor's models, the second, i.e., that concerning the Phillips Curve, refers to Rose's model (1967). Indeed, in the latter model, the relation between the employment rate (z) and the rate of growth of money wages (\dot{w}/w) is described by $\dot{w}/w = F(z)$ where the nonlinear function F is assumed to have the following shape:



The reasons for such a shape are explained by Rose in the following passage:

"...The percentage rate of wage inflation is an increasing function of z "; however, "...the relationship just described is nonlinear. In some neighbourhood of the z at which unemployment is balanced by unfilled vacancies frictions and imperfections weaken the responsiveness of wage inflation to changes in z ; but at sufficiently high and low values of z the pressures of competition overcome inertia, so that a change in z brings about a large change in wage inflation".

(Rose, 1967, p.158).

function. Then, in Section 5.3., the same problem is analysed with regard to Kaldor's 1940 model.

5.2. Goodwin's Model

Goodwin's starting point, in his 1951 article, is a criticism of "standard" linear models of the business cycle in which the resulting fluctuations in national income either explode or die out. To overcome this difficulty which is inescapable in a linear model, he introduces some modifications of the "standard" model of the multiplier-accelerator type¹, the result of which is the following non-linear equation:

$$\epsilon \ddot{y} + \psi(\dot{y}) + (1 - c)y = 0 \quad (1),$$

where y is the deviation of income from the equilibrium level; $\dot{y} = dy/dt$, and $\ddot{y} = d^2y/dt^2$.

Equation (1), which describes a stable limit cycle, is the result of a "step by step" procedure by means of which Goodwin introduces more and more modifications of the linear model.

1. See, for instance, Samuelson, 1939.

2. As we have seen in Chapter 4.1, (1) is an equation of Lord Rayleigh's rather than of van der Pol's type.

In a simple continuous model with a linear version of the acceleration principle and a consumption function of the type we will consider below, the dynamics of the economy is described by a linear differential equation of the first order. On the contrary, the first step of Goodwin's procedure, which consists in making less rigid the strict proportionality implied by the acceleration principle according to which actual capital stock is always maintained at the desired relation with output, gives rise to a very different picture of the dynamics of the system which would not be obtainable as a solution for a linear model¹. Indeed, this results in fluctuations in income which "maintain themselves", i.e., which are endogenous and persistent and which, moreover, are independent of the initial conditions. Given the oversimplifications involved in this first version of the model, however, and above all given the fact that the discontinuous formulation of the nonlinear investment function leaves little room for an analytical solution, we prefer to study to a larger and

1. In the simplest version of Goodwin model, it is assumed that the production of investment goods can only take place "... in either of two extreme ways, capacity output of investment goods or zero gross investment" (Goodwin, 1951, p.5). As a consequence, the relation which is assumed to hold between investment and the difference between the desired capital stock $-k^d -$ and the actual capital stock $-k -$ is discontinuous: when $k^d - k \geq 0$, investment proceeds at the capacity level whereas when $k^d - k < 0$, net investment is negative and equal to total depreciation. For a description of the resulting dynamics, see Goodwin, 1951, pp.4-7 and Gandolfo, 1973, pp.432-435.

deeper extent the models resulting from the successive steps of Goodwin's "step by step-procedure".

Firstly, Goodwin introduces the dynamic operation of the multiplier by considering the following equation:

$$Y = C + I - \epsilon \dot{Y} \quad (1),$$

instead of the usual formulation according to which:

$$Y = C + I \quad (1').$$

As stressed by Goodwin¹, in equation (1) the fact that the process of multiplication takes time is taken into account. Indeed, inserting in (1) a consumption function of the type:

$$C = C_0 + cY \quad (2),$$

we obtain:

$$\dot{Y} = \left(\frac{s}{\epsilon}\right) \left\{ \frac{I + C_0}{s} - Y \right\},$$

i.e., Y adjusts smoothly to the deficiency $\left\{ \left(\frac{I + C_0}{s}\right) - y \right\}$ with a speed of adjustment equal to $\left(\frac{s}{\epsilon}\right)$ - rather than instantaneously.

1. See Goodwin, 1951, p.235.

Secondly, Goodwin assumes that investment consists of an autonomous part, $l(t)$, and an induced part, $\phi(\hat{Y})$, which is a continuous nonlinear function such that:

$$\frac{d \phi(\hat{Y})}{d \hat{Y}} = v(>0),$$

for values of \hat{Y} in the middle range, whereas

$$\lim_{\hat{Y} \rightarrow \pm\infty} \frac{d \phi(\hat{Y})}{d \hat{Y}} = 0^1.$$

Thus, we can write equation (1) as:

$$\begin{aligned} Y &= \frac{I + C_0 - \epsilon \hat{Y}}{s} = \\ &= \frac{l(t) + \phi(\hat{Y}) + C_0 - \epsilon \hat{Y}}{s}, \end{aligned}$$

i.e.:

1. Investment is also assumed to have an upper limit - equal to I^ - and a lower limit - equal to $-I^*$ - where I^* is a positive constant corresponding to the maximum rate of investment allowed by existing productive capacity and $-I^*$, a negative constant corresponding to zero gross investment. Following Tinbergen (1944, p.24), we can explain the continuity of the function saying that not all firms reach their capacity limit and the level of investment corresponding to zero gross investment at the very same moment.

$$Y = \frac{I(t) + C_0 + \Gamma(\hat{Y})}{s} \quad (3),$$

where:

$$\Gamma(\hat{Y}) = \phi(\hat{Y}) - \epsilon\hat{Y}.$$

At the equilibrium point, we have:

$$\phi(0) = v\hat{Y},$$

so that, linearizing equation (3) around it, we obtain:

$$sY = I(t) + C_0 + (v - \epsilon)\hat{Y}.$$

Thus, ignoring for the moment the autonomous part of investment and of consumption, we can say that the equilibrium is locally stable or locally unstable according to whether:

$$\frac{s}{v - \epsilon} < 0,$$

or:

$$\frac{s}{v - \epsilon} > 0.$$

As a consequence, assuming that $v > \varepsilon$, as Goodwin does¹, the equilibrium point is locally unstable. In this case, the system may well "explode" beyond the region of valid linear approximation. Thus, the local stability analysis does not allow us to draw conclusions about the global behaviour of the system which, obviously, depends crucially on the nonlinearity of the investment function².

Going back to the original nonlinear equation (3), with $l(t) + C_0 = 0$, and differentiating with respect to t both sides, we obtain:

$$s\dot{Y} = \{\phi'(\dot{Y}) - \varepsilon\}\ddot{Y},$$

i.e.:

$$\begin{aligned}\dot{x} &= \left\{ \frac{s}{\phi'(x) - \varepsilon} \right\} x = \\ &= \left\{ \frac{s}{\Gamma'(x)} \right\} x\end{aligned}\quad (4),$$

where $x = \dot{Y} = dY/dt$.

The curve $\Gamma'(x)$ can easily be obtained by drawing in the same graph the curve $\phi'(x)$ and the horizontal straight line of ordinate

1. See Goodwin, 1951, p.9.

2. For example, in a nonlinear model, there is an additional concept of stable equilibrium, namely, that of stable motion as represented by a stable limit cycle to which all trajectories tend. See Chiarella, 1986, p.289.

ε as is shown in Figure 5.1.a. Taking for all x the difference between $\phi'(x)$ and ε , which is equal to zero both for the negative value $x = x_-$ and for the positive value $x = x_+$ and has a maximum equal to $(v - \varepsilon)$ for all values of x in the range (x_1, x_2) , we obtain the picture given in Figure 5.1.b.

As we see, the analysis of the original nonlinear equation provides us with insights into the dynamics of the system that the local stability analysis could not provide. Indeed, although the equilibrium point is unstable, and $\phi'(x) \geq 0$ in the interval $x_- \leq x \leq x_+$, we have $\phi'(x) < 0$ in the intervals $x < x_-$ and $x > x_+$. Thus, although for values of $|x|$ "near" the equilibrium point $\phi'(x)$ is positive so that x moves away from its equilibrium value, this cannot lead to an "explosion" because for values of $|x|$ "far" from the equilibrium point $\phi'(x)$ is negative so that x gets nearer to its equilibrium value¹.

We can foresee what are the consequences for the dynamics of the system of the nonlinearity of the function $\phi(x)$ if we consider its representation in the $(Y, \dot{Y}) = (Y, x)$ phase plane.

To this end, let us note that equation (3) implicitly describes a function of the type:

$$x = \dot{Y} = h(Y) \tag{5},$$

1. As we will see, however, it would be wrong to draw the conclusion from this that the time paths of x tend either to x_- or x_+ ; indeed, the latter are not equilibrium points of equation (4). Cfr. Chiarella, 1986, pp.286-289.

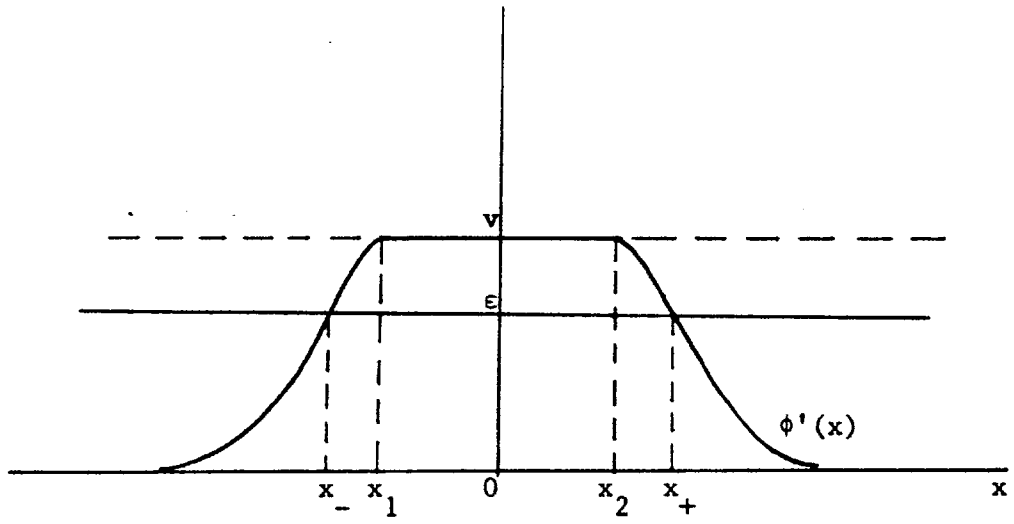


Fig.5.1.a

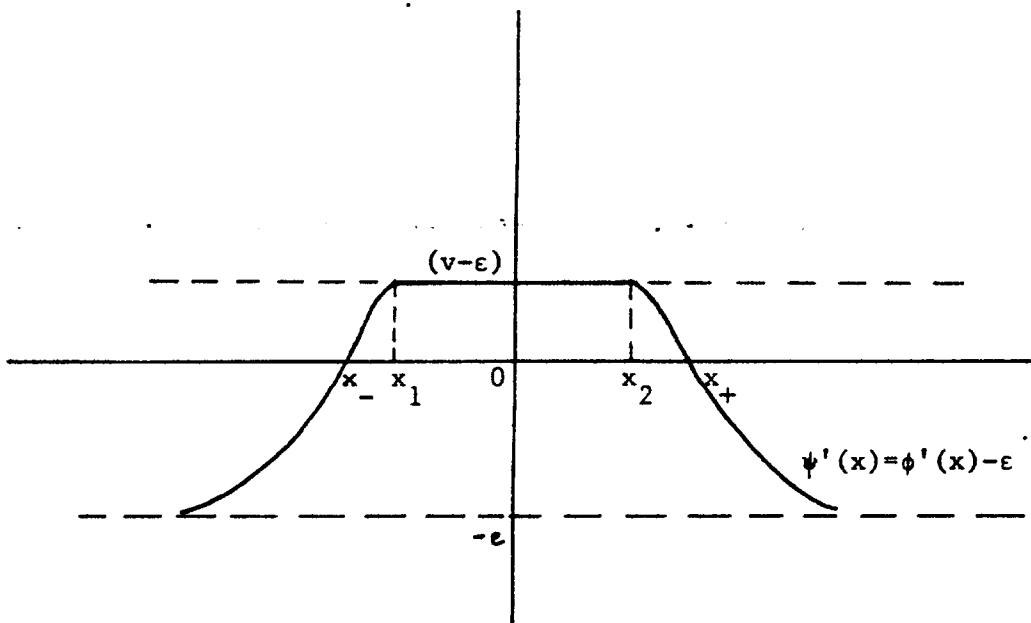


Fig.5.1.b

the derivate of which can be easily calculated by using the Implicit Function Rule¹. Taking the total differential of both sides of (3), we obtain:

$$\{\phi'(x) - \epsilon\}dx - sdy = 0 = \Gamma'(x)dx - sdy,$$

1. Following, for example, Chiang (1984, p.206), we can state the theorem for our two dimensional case as follows:

Theorem: Given an equation in the form of:

$$H(x,y) = 0 \quad (a),$$

if:

- (i) the function H has continuous partial derivatives H_x, H_y ;
- (ii) at a point (x_0, y_0) satisfying the equation (a), H_x is nonzero;

then, there exists a one-dimensional neighborhood of y in which x is an implicitly defined function of the variable y in the form of $x = h(y)$. This implicit function satisfies $x_0 = h(y_0)$. It also satisfies the equation (a) for every y_0 in the neighborhood - thereby giving (a) the status of an identity in that neighborhood. Moreover, the implicit function is continuous, and has continuous derivative $h'(y) = dh/dy$.

In our case, we have:

$$H(x,Y) = \phi(x) - \epsilon x - sY = 0,$$

from which:

$$H_x = \phi'(x) - \epsilon,$$

$$H_y = -s.$$

Thus, given condition (ii) of the Theorem, and given that $H_x = 0$ for $x = x_-$ and $x = x_+$, equation (5) is not defined for the latter two values of the variable x .

so that:

$$\frac{dx}{dy} = h'(y) = \frac{s}{\{\phi'(x) - \varepsilon\}} = \frac{s}{\Gamma'(x)}.$$

Given the picture we have obtained in Figure 5.1, we know that:

$$\frac{dx}{dy} \left\{ \begin{array}{l} > 0, \text{ for } x_- < x < x_+ \\ < 0, \text{ for } x < x_- \text{ and } x > x_+ \end{array} \right. ,$$

with:

$$\frac{dx}{dy} = \frac{s}{v - \varepsilon} \text{ for } x_1 \leq x \leq x_2.$$

Thus, we obtain the picture of the dynamics of the system shown in Figure 5.2.

The motion of the system is along AB and CD with the direction as indicated by the arrows. However, at B, where $x = x_+$, and at D, where $x = x_-$ and hence the dynamics of the system is not represented by equation (5), we have points of discontinuity for which x undergoes a discontinuous change (from B to C and from D to A). As a result, around the unstable equilibrium point E there is the limit cycle ABCD in which the discontinuities - or "jumps" - may be considered as approximations to rapid changes¹.

1. The dynamics of the model with a dynamical multiplier and a nonlinear accelerator is discussed by Goodwin, 1951, pp.8-11.

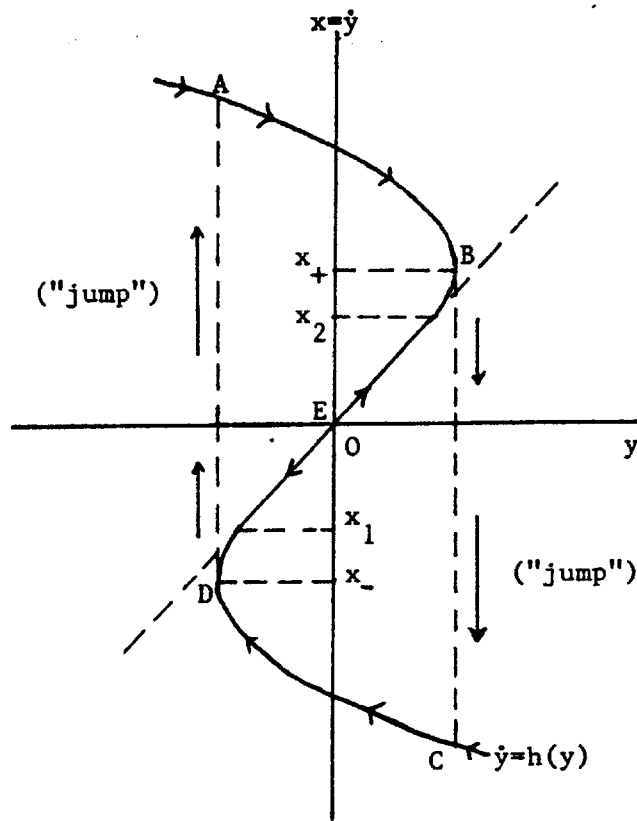


Fig.5.2

Finally, Goodwin introduces a time lag θ between decisions to invest - O_d - and the resulting outlays - O_I - such that:

$$O_I(t+\theta) \doteq O_d(t) = \phi(\hat{Y}(t)).$$

Thus:

$$I(t+\theta) = l(t+\theta) + O_I(t+\theta) = l(t+\theta) + \phi(\hat{Y}(t)),$$

and, inserting this expression for I in (1), we obtain the mixed differential-difference equation which describes the dynamics of the system. However, as an approximation - which is obtained expanding the two leading terms in a Taylor series and dropping all but the first two terms in each¹ - Goodwin considers the following nonlinear differential equation of the second order which is then analysed with regard to both cyclical and stability properties:

$$\epsilon \theta \ddot{Y} + \{\epsilon + (1 - c)\theta\}\dot{Y} - \phi(\hat{Y}) + (1 - c)Y = l(t+\theta) \quad (6).$$

Although Goodwin obtains equation (6) as an approximation to the final equation of his "step by step" procedure, it is possible to obtain the same equation in a more direct way². Indeed, by

1. See Goodwin, 1951, p.12.

2. See Allen, 1967, pp.378-384. See also Medio, 1979, pp.66-68.

analogy with equation (1), where income is assumed to adjust smoothly to the difference between demand and income with a speed of response equal to $(1/\varepsilon)$, we can assume that the level of induced investment adjusts smoothly to the desired level - $\phi(\hat{Y})$ - with a speed of adjustment equal to $(1/\theta)$ and write:

$$\frac{dY}{dt} = \hat{Y} = \left(\frac{1}{\varepsilon}\right)\{(C + I) - Y\},$$

$$\frac{d\hat{K}}{dt} = \hat{I} = \left(\frac{1}{\theta}\right)\{\phi(\hat{Y}) - [I - l(t+\theta)]\},$$

or:

$$(\varepsilon D + 1)Y = C + I \quad (7),$$

$$(\theta D + 1)I = \phi(\hat{Y}) + l(t+\theta) \quad (8),$$

where $D = d/dt$, from which we obtain a second order equation identical to equation (6).

To simplify, Goodwin assumes that the autonomous expenditure $l(t+\theta)$ - the "forcing" term - is constant and equal to O^* . This means that, if we choose the time unit such that $\theta = 1$, the dynamics of the deviations of income from the equilibrium level $\bar{Y} = O^*/s$ is described by the following equation:

$$\varepsilon \ddot{y} + \psi(\dot{y}) + sy = 0 \quad (9),$$

where $y = Y - \bar{Y}$ and $\psi(\dot{y}) = (\varepsilon + s)\dot{y} - \phi(\dot{Y})$.

The local properties of equation (9) - or, equivalently of the dynamical system (7)-(8) - can be analysed by linearizing the equation around its equilibrium point. For example, linearizing around the equilibrium point $(y, I) = (0, 0)$ the homogenous system, given that the slope of the nonlinear investment function at $\dot{y} = 0$ is equal to v , we obtain:

$$\dot{y} = \left(\frac{1}{\epsilon}\right)(c - 1)y + \left(\frac{1}{\epsilon}\right)I \quad (7'),$$

$$\dot{I} = v\dot{y} - I \quad (8'),$$

or, in matrix notation:

$$\begin{bmatrix} \dot{y} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} -\frac{s}{\epsilon} & \frac{1}{\epsilon} \\ \frac{-vs}{\epsilon} & \frac{v}{\epsilon} - 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ I \end{bmatrix} = A \begin{bmatrix} y \\ I \end{bmatrix}.$$

The matrix A is such that:

$$\text{Det}(A) = \frac{s}{\epsilon} > 0,$$

and:

$$\text{Tr}(A) = -\frac{s}{\epsilon} + \frac{v}{\epsilon} - 1 \gtrless 0 \longleftrightarrow v \gtrless s + \epsilon.$$

Thus, if:

$$v > s + \epsilon,$$

$\text{Tr}(\mathbf{A})$ is positive and the equilibrium is locally unstable. As a consequence, to draw conclusions about the global behaviour of the system, we must analyse the nonlinear equation (9) directly.

To do this, Goodwin makes use of Liénard's method of graphical integration which we have described in Chapter 4.1 and shows that, in the case in which the above condition holds, so that the equilibrium point is unstable, equation (9) describes a stable limit cycle¹. For example, the resulting dynamics of the economy

1. To apply Poincaré-Liénard's method of graphical integration, Goodwin, first of all, must reduce the equation to a dimensionless form. To this end, he takes $\sqrt{\epsilon/s}$ as new unit of time such that:

$$t = \sqrt{\frac{\epsilon}{s}} \cdot t_1.$$

Moreover, he takes \dot{y}_0 as any convenient unit in which to measure velocity such that:

$$\dot{y} = \dot{y}_0 \dot{x} = \dot{y}_0 \frac{dx}{dt_1} = \frac{dy}{\sqrt{(\epsilon/s)} dt_1},$$

from which:

$$y = \dot{y}_0 \sqrt{\frac{\epsilon}{s}} \cdot x,$$

and:

$$\ddot{y} = \frac{d^2 y}{dt^2} = \frac{d}{dt} \dot{y}_0 \dot{x} = \dot{y}_0 \frac{d^2 x}{dt dt_1} = \dot{y}_0 \sqrt{\frac{s}{\epsilon}} \cdot x.$$

Thus, inserting in (9), we obtain:

$$\ddot{x} + X(\dot{x}) + x = 0,$$

where:

$$X(\dot{x}) = \frac{\psi(\dot{y}_0 \dot{x})}{\dot{y}_0 \sqrt{\epsilon s}}.$$

(Footnote continues on next page)

is as shown in Figure 5.3¹.

(Footnote continued from previous page)

Alternatively, we can write:

$$v(dv/dx) + X(v) + x = 0,$$

where $v = \dot{x}$. Given the equation written in this form, it is possible to apply Liénard's method and show that all trajectories tend towards a stable limit cycle. See Goodwin, 1951, pp.11-15. See also Le Corbeiller, 1936, p.236.

1. The graph has been drawn using the program to simulate with differential equations called PHASER. To do this, it has been necessary to specify a form for the nonlinear investment equation $\phi(\dot{y})$. Following Allen, and given that the investment function must satisfy four conditions, we have chosen a form of the type:

$$\phi(\dot{y}) = \left\{ \frac{a}{be^{-cy} + 1} - 1 \right\} d,$$

where the four coefficients a , b , c , and d are determined as solutions of the following system:

- (1) $\phi(0) = 0$,
- (2) $\phi(-\infty) = -I^{**}$,
- (3) $\phi(+\infty) = I^*$,
- (4) $\phi'(0) = v$.

Following this procedure, we have obtained the following specification for the investment function:

(Footnote continues on next page)

Given this result, Goodwin's conclusion is that:

"...making only assumptions acceptable to most business cycle theorists, along with two simple approximations, we have been able to arrive at a stable, cyclical motion which is self-generating and self-perpetuating"

(Goodwin, 1951, p.12).

However, in order to understand the "robustness" of this result, and to evaluate the relevance of Benassy's criticism, we now consider the problem of the role that is played in Goodwin's model by the choice of the shape of the investment function.

To this end, let us write equation (9) in its equivalent van der Pol's form:

$$\ddot{x} + \psi'(x)\dot{x} + sx = 0 \quad (10),$$

where $x = \dot{y}$.

Given that the slope of Goodwin's nonlinear investment function is known, the curve $\psi'(x) = (\varepsilon + s) - \phi'(x)$ can easily be obtained

(Footnote continued from previous page)

$$\phi(\dot{y}) = \left\{ \frac{I^* + I^{**}}{I^* \exp(-(I^* + I^{**})/(I^* I^{**})v\dot{y})) + I^{**}} - 1 \right\} I^{**},$$

which we have then employed for the computer simulation. See Allen, 1967, pp.378-380.

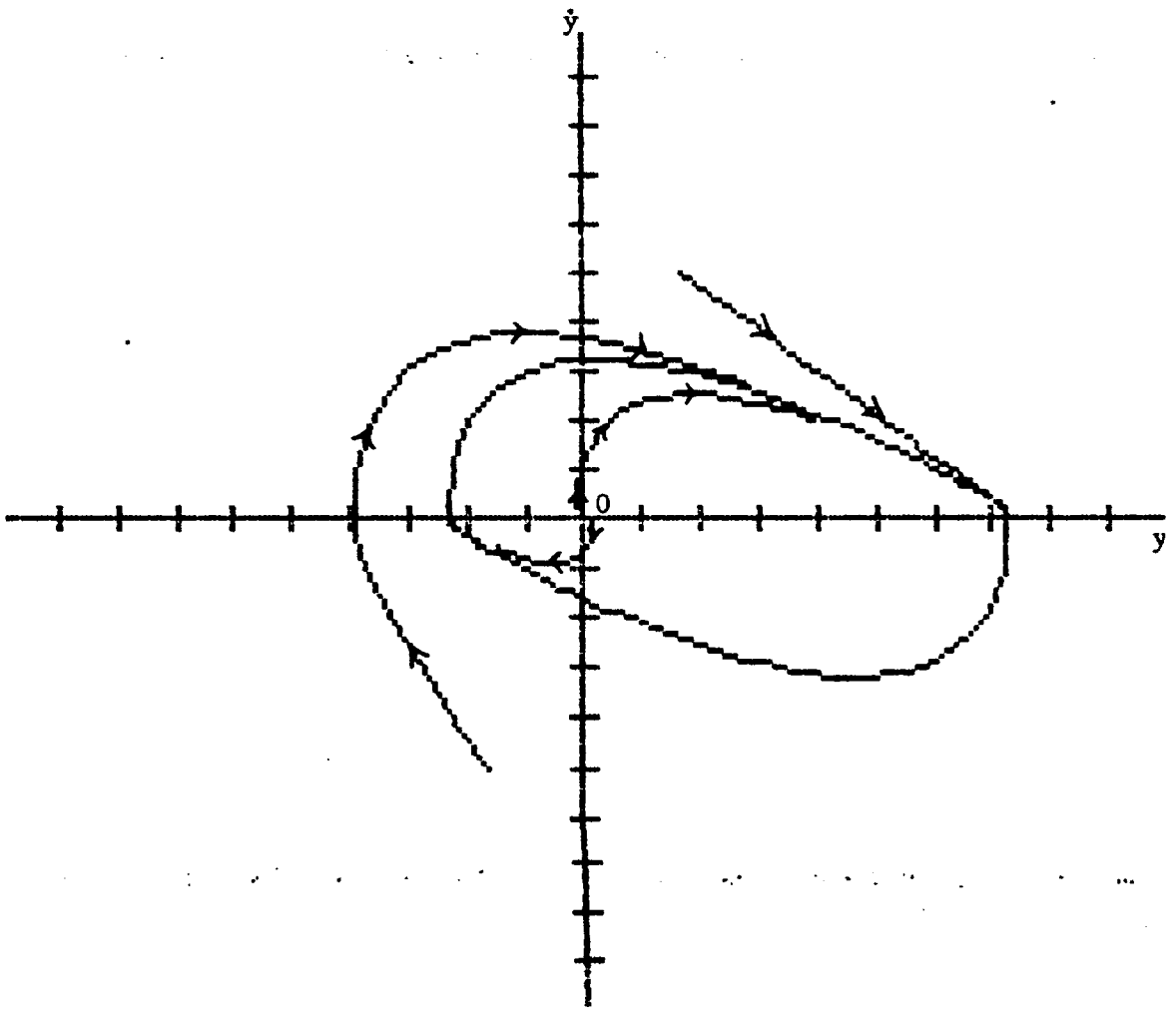


Fig.5.3

as is shown in Figure 5.4¹.

Intuitively, as stressed by Ichimura (1954, pp.202-203), it is possible to expect a self-sustaining oscillatory movement for this dynamic system. Indeed, when $|x|$ is "small", the function $f'(x)$, representing the dampening which operates in the system, is negative so that x increases whereas when $|x|$ is "large", the function $f'(x)$ is positive so that x decreases. However, it must be rigorously proved that the result of these counteracting effects is a limit cycle.

As we have seen in the previous chapter, this can be done in an illuminating way by applying Liénard's criterion for the existence of periodic solutions. To introduce the use of this method for the more general case we are now analysing, let us consider the following equation of generalized van der Pol's type:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad (11),$$

where $f(x)$ is an even function such that, writing $F(x)$ for the odd function $\int_0^x f(t)dt$, there exists an $x = x_0$ such that $F(x) < 0$ for $0 < x < x_0$; $F(x) > 0$, $F'(x) > 0$ for $x > x_0$ with $\lim_{x \rightarrow +\infty} F(x) = +\infty$; and where $g(x)$ is differentiable and such that $g(x) \geq 0$ for $x \geq 0$.

Writing (11) in the Liénard plane, we have:

1. We are considering the case in which the local stability condition is not satisfied, namely, the case in which we have:

$$v = \phi'(0) > \varepsilon + s.$$

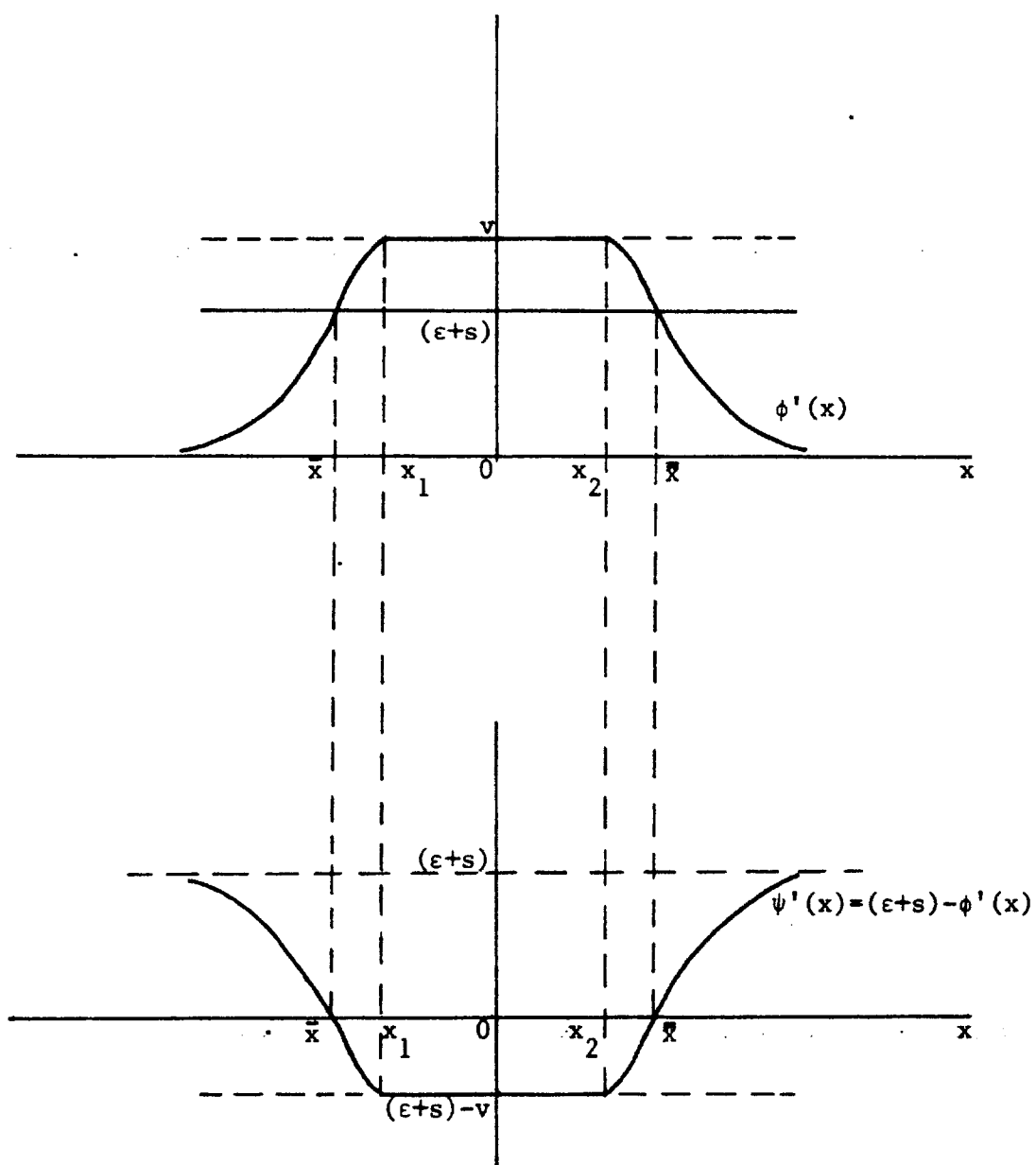


Fig.5.4

$$\dot{x} = y - F(x),$$

$$\dot{y} = \dot{x} + f(x)\dot{x} = -g(x),$$

so that, by analogy with the simpler case we have analysed in the previous chapter, we can write:

$$E = \frac{y^2}{2} + G(x),$$

where E is the total energy and $G(x) = \int_0^x g(t)dt$.

Thus, the exchange of energy at time t is equal to:

$$\frac{dE}{dt} = F(x)\frac{dy}{dt},$$

and the limit cycle (sufficient) condition becomes:

$$\oint dE = \oint F(x)dy = 0 \quad (12).$$

As shown by Levinson and Smith¹, under the assumptions above made about the functions $f(x)$, $F(x)$, and $g(x)$, this is true at the most once, implying that equation (11) possesses a unique periodic solution.

For Goodwin's model, where

1. See Levinson-Smith, 1942, pp.389-403.

$$f(x) = \psi'(x),$$

$$g(x) = sx,$$

we find:

$$F(x) = (\epsilon + s)x - \phi(x).$$

As shown in Figure 5.5, there exists an $x = x_0$ such that the curve $F(x)$ satisfies the above properties; moreover, $g(x) = sx$ is such that $g(x) \geq 0$ for $x \geq 0$. As a consequence, there exists a unique limit cycle the amplitude of which is given by the solution of (12).

Thus, Liénard's criterion for the existence of a limit cycle clarifies the role played in Goodwin's model by the specification of the nonlinear investment function. Indeed, the latter is one of the functions we have considered in Figure 5.5 to determine the shape of the curve $F(x)$. In particular, it is important to notice that, given the sigmoid shape of the investment function, there are three points of intersection between the latter function and the straight line of slope $(\epsilon + s)$.

To better understand the use of this criterion, however, it is useful to show how it can be applied also to other nonlinear models of the business cycle. For example, we can show how it can be used to determine the (sufficient) condition for the existence of a limit cycle in Kaldor's 1940 model.

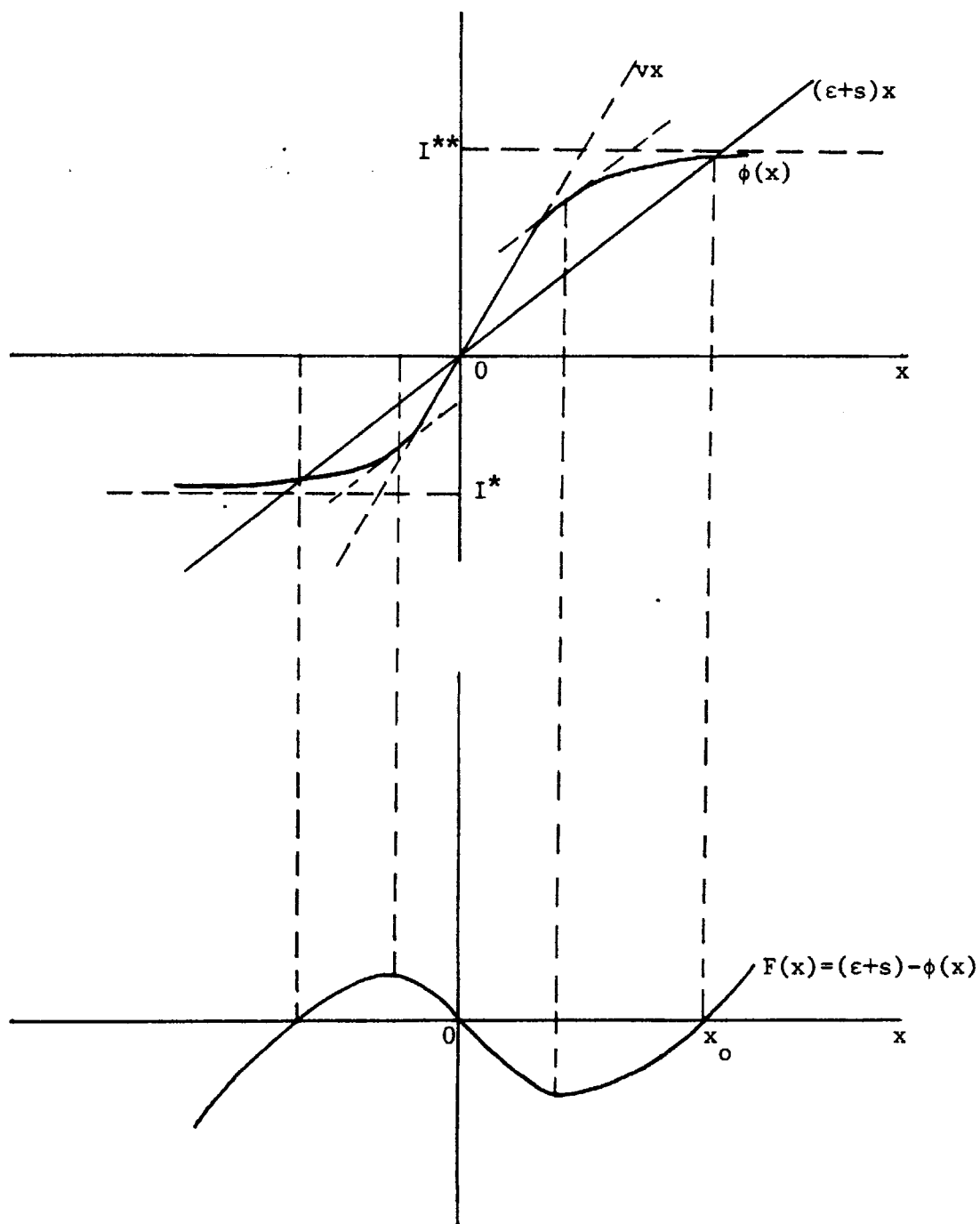


Fig.5.5

5.2. Kaldors' Model

In Kaldor's 1940 model¹, ex-ante investment and saving are assumed to be functions of income and the capital stock,

$$I(y,k) \text{ and } S(y,k),$$

such that:

$$I_y, S_y, S_k > 0 \text{ and } I_k < 0.$$

To obtain the system which describes the dynamics of the economy, it is enough to take account of the fact that income rises every time that ex-ante investment is greater than ex-ante saving. Thus, assuming that the adjustment takes place at a speed equal to $(1/\mu)$, we can write:

$$\mu \dot{y} = I(y,k) - S(y,k) \quad (1).$$

On the other hand, the equation governing the accumulation or decumulation of capital can have different forms according as to whether it is assumed that ex-ante investment, ex-ante saving, or a

1. The basic reference for a mathematical analysis of Kaldor's model is Chang-Smyth, 1971.

weighted average of the two is realized. Choosing the latter assumption, which is the more general, we have:

$$\dot{k} = \delta I(y, k) + (1 - \delta) S(y, k) \quad (2),$$

where $0 \leq \delta \leq 1$ ¹.

The two functions $I(y, k)$ and $S(y, k)$ are assumed to be nonlinear functions of y such that:

$$I_y > S_y,$$

for normal levels of income, and:

$$I_y < S_y,$$

for abnormally high or abnormally low levels of income. Moreover, it is assumed that the two functions are such that the stationary-state equilibrium (\bar{y}, \bar{k}) occurs for a normal level of income².

To apply Liénard criterion for the existence of periodic solutions, we must reduce the system (1)-(2) to a nonlinear

1. Different authors have used different formulations. For example, in Chang-Smith, 1971, it is assumed that, in (2), $\delta = 1$, whereas in Ichimura, 1954, that $\delta = 0$. For an analysis of the more general case we have chosen to consider, see Kosobud-O'Neil, 1972, pp.73-76.

2. See Kaldor, 1940; Chang, Smyth, 1971, p.40.

differential equation of van der Pol's type. To do this, we note that equation (1) implicitly define a relation of the type:

$$k = \psi(y, \dot{y}) \quad (3).$$

with:

$$\frac{\partial k}{\partial y} = \psi_y = \frac{S_y - I_y}{I_k - S_k}, \quad \frac{\partial k}{\partial \dot{y}} = \psi_{\dot{y}} = \frac{\mu}{I_k - S_k} \cdot 1.$$

Thus, inserting (3) in (2), we obtain the required second order equation:

$$\psi_y \dot{y} + \psi_{\dot{y}} \ddot{y} = \delta I(y, k) + (1 - \delta) S(y, k) \quad (4).$$

To simplify, we assume, following Ichimura (1954, p.214), that $I(y, k)$ and $S(y, k)$ are of the form:

$$I(y, k) = f(y) - mk \quad (5),$$

$$\begin{aligned} S(y, k) &= g(y) + nk = \\ &= sy + nk \end{aligned} \quad (6),$$

1. These two partial derivatives of the function $\psi(y, \dot{y})$ can be easily calculated by employing the Implicit Function Rule.

where $I_y = f'(y) > 0$, $I_k = -m < 0$, $S_y = s > 0$, and $S_k = n > 0$ and where, to maintain a closer relation to Goodwin's model, we have assumed that the saving function is linear both in y and in k^1 .

In this case, Kaldor's dynamical system becomes:

$$\mu \dot{y} = \{f(y) - sy\} - (m + n)k \quad (1'),$$

$$\dot{k} = \delta\{f(y) - mk\} + (1 - \delta)\{sy + nk\} \quad (2'),$$

where:

$$k = \psi(y, \dot{y}) = \frac{1}{m + n} \{f(y) - sy - \mu \dot{y}\} \quad (3'),$$

such that:

$$\psi_y = \frac{1}{m + n} \{f'(y) - s\}$$

and:

1. The possibility of taking a linear saving function seems to be suggested by Kaldor himself when he writes:

"...our analysis would remain valid if only one of the two functions behaved in the manner suggested, while the other was linear"

(Kaldor, 1940, p.82).

In the Appendix to the 1940 article, Kaldor himself, in order to compare his theory with Kalecki's theory (see Kalecki, 1937), considers a linear savings function.

$$\psi_{\bar{y}} = - \frac{\mu}{m + n}.$$

Before writing down equation (4) for the simplified case which we are now considering, we analyse the (local) stability properties of this dynamical system linearizing it around the equilibrium point (\bar{y}, \bar{k}) .

Doing this, we obtain a system of the type:

$$\dot{X} = AX,$$

where:

$$X = \begin{bmatrix} y - \bar{y} \\ k - \bar{k} \end{bmatrix}, \quad A = \begin{bmatrix} (\frac{1}{\mu})[f'(\bar{y}) - s] & -(\frac{1}{\mu})(m + n) \\ \delta f'(\bar{y}) + s(1 - \delta) & n(1 - \delta) - \delta m \end{bmatrix}.$$

Thus, the characteristic equation is:

$$\lambda^2 - [\text{Tr}(A)]\lambda + \text{Det}(A) = 0,$$

where:

$$\text{Tr}(A) = (\frac{1}{\mu})[f'(\bar{y}) - s] + n(1 - \delta) - \delta m \gtrless 0,$$

$$\text{Det}(A) = (\frac{1}{\mu})[nf'(\bar{y}) + ms] > 0.$$

Given that $\text{Det}(A)$ is always positive, stability is determined by the sign of $\text{Tr}(A)$. In particular, if:

$$\text{Tr}(A) > 0 \iff \left(\frac{1}{\mu}\right)[f'(\bar{y}) - s] + n(1 - \delta) - \delta m > 0 \quad (7),$$

the equilibrium point is unstable.

In the case in which condition (7) holds, it is possible to show, by applying the same Liénard criterion we have above applied to Goodwin's model, that Kaldor's dynamical system describes persistent fluctuations of the income level.

To apply this criterion, we consider equation (4) which, for the case we are now analysing, becomes:

$$\ddot{y} + f_1(y)\dot{y} + g_1(y) = 0 \quad (8),$$

where:

$$f_1(y) = \left(\frac{1}{\mu}\right)\{s - \mu[n - \delta(m + n)] - f'(y)\},$$

$$g_1(y) = \left(\frac{1}{\mu}\right)\{msy + nf(y)\}.$$

As was the case for Goodwin's model, we obtain a differential equation of (generalized) van der Pol's type. Taking account of the assumptions made by Kaldor regarding the slopes of the investment and saving functions, it is possible to obtain the shape of the nonlinear damping as is shown in Figure 5.6. As we see, for values of y "near" the equilibrium point, which is in the range of "normal" values of income, the damping is negative so that y moves

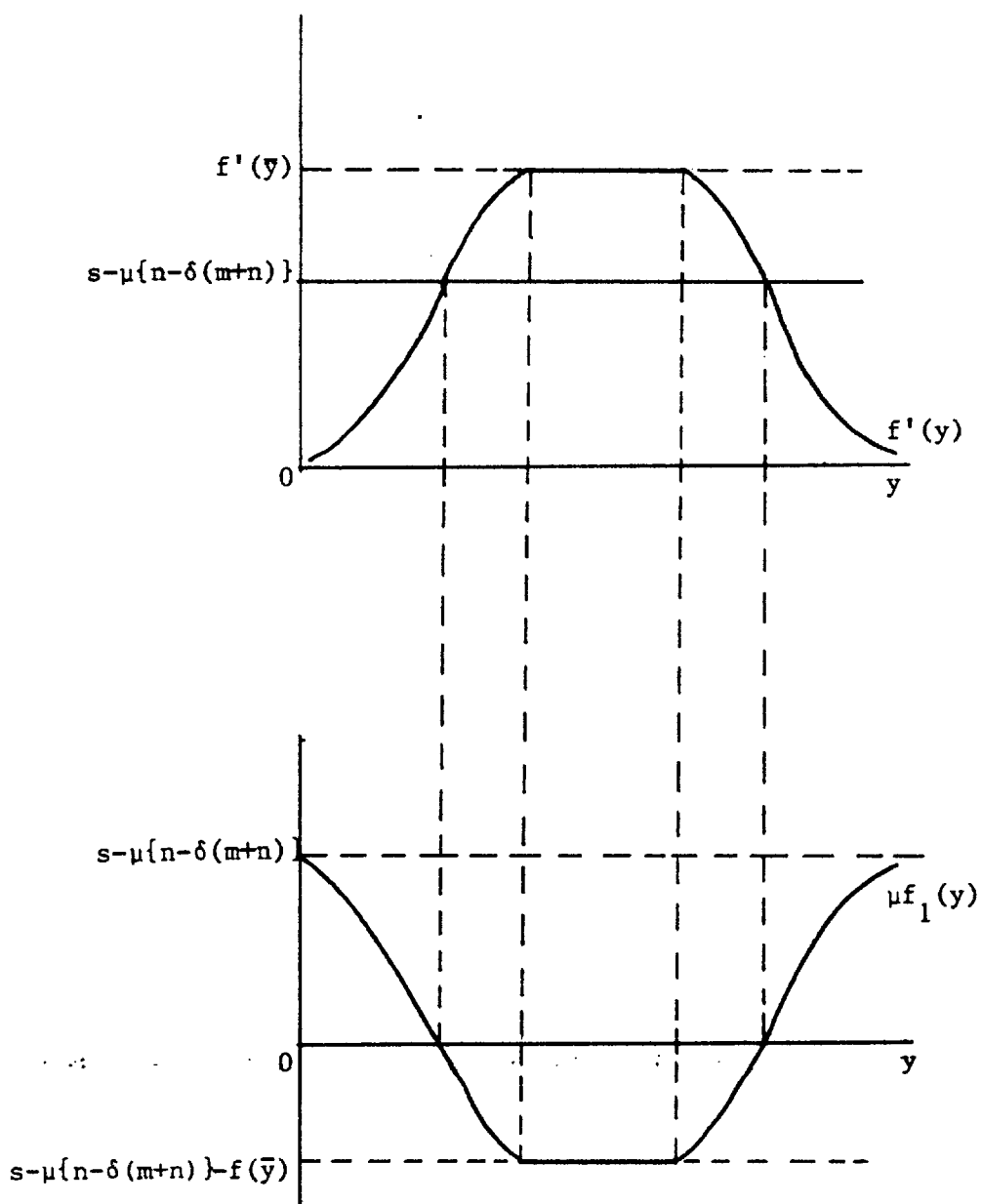


Fig.5.6

away from its equilibrium value. On the other hand, for values of y "far" from the equilibrium point, the damping is positive so that y gets nearer to its equilibrium value. By analogy with the solution of a linear differential equations of the second order with constant coefficients, we can expect that the interplay of these counteracting effects results in a steady oscillation of a certain amplitude.

Applying Liénard's criterion, it is possible to show that this is in fact the case. Indeed, assuming that $\{s - \mu[n - \delta(m + n)]\} < f'(\bar{y})$, which, as we have above seen, implies that the equilibrium is unstable, we obtain the graph of the function $F_1(y)$ which is shown in Figure 5.7. As we see, the function $F_1(y)$, when the values of y are taken as deviations from \bar{y} , has the same shape as in Goodwin's model and plays the same role with regard to the generation of the limit cycle the amplitude of which is determined by the solution of the equation:

$$\oint F_1(x) dy = 0,$$

solution which exists and is unique. In Kaldor's model, therefore, the choice of the shape of the nonlinear investment function, which contributes to the determination of the shape of the function $F_1(x)$, also appears to be crucial.

To conclude the analysis of these two models belonging within the same framework, it is interesting to note that, although Kaldor and Goodwin consider investment functions with the same shape, their economic content is very different. However, this fact is not

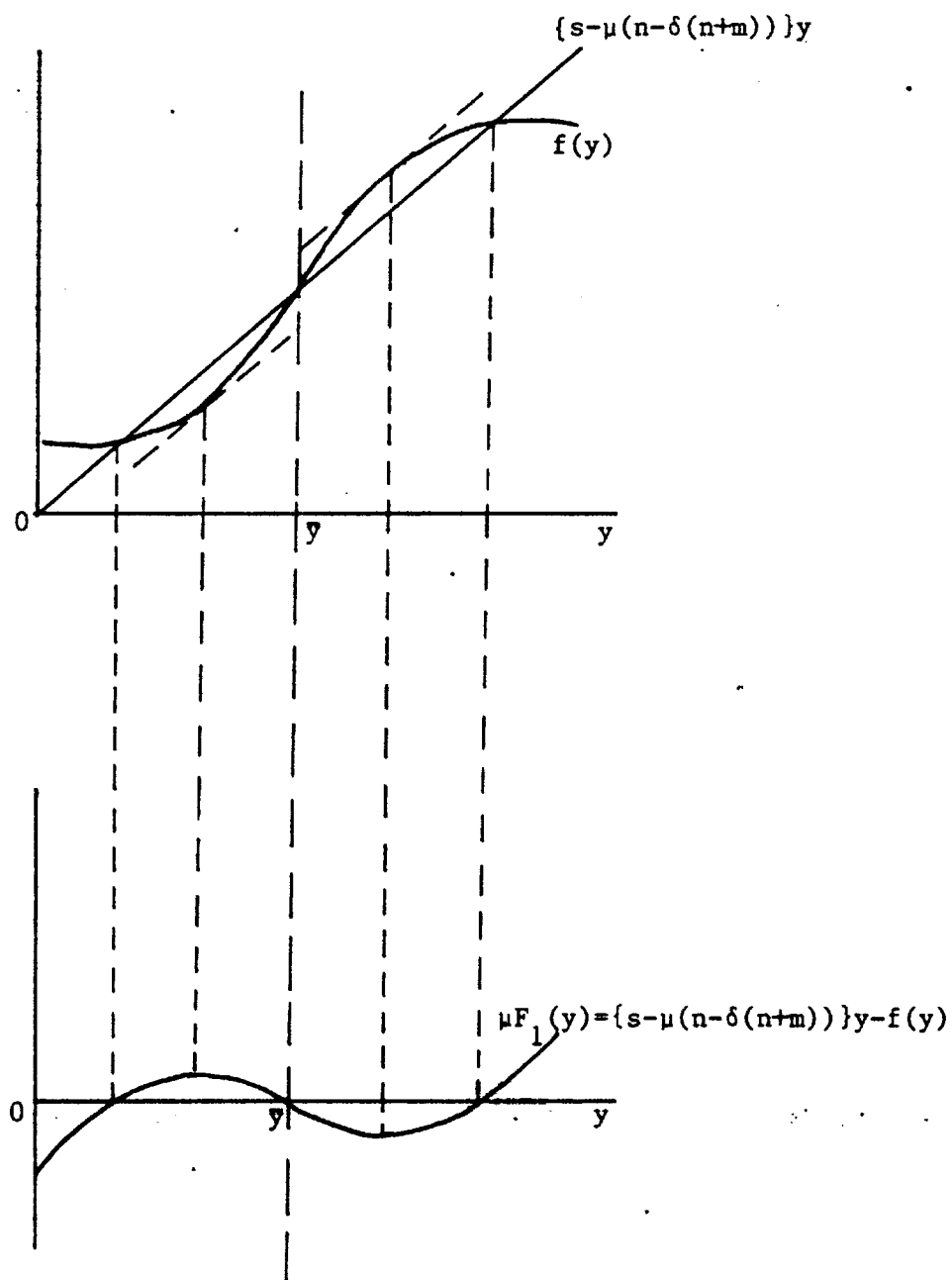


Fig.5.7

sufficiently highlighted by a purely mathematical analysis of the two models based on the Liénard's criterion which, as we have seen, leads to almost identical conclusions for the two models.

To understand the different economic content of the two nonlinear investment functions, it is necessary to analyse somewhat more deeply the way in which they are introduced by the two authors.

In Kaldor's original formulation, short-period investment depends positively on the level of economic activity (x), i.e.:

$$I = I(x),$$

with $I'(x) = dI/dx > 0$ ¹. In particular, to understand the different economic content of Kaldor's and Goodwin's investment functions, it is enough to note that in Kaldor's formulation, investment depends on the level rather than on the rate of change of the level of economic activity². This relation, as we have

1. As stressed by Kaldor, this simply means that "... the demand for capital goods will be greater the greater the level of production" (Kaldor, 1940, p.79).

2. Kaldor explicitly counterposes his assumption to the accelerator principle. Indeed, in his opinion, the assumption made by him:

"... should not be confused with the 'accelerator principle' (of Prof. J.M. Clark and others), which asserts that the demand for capital goods is a function of the rate of change of the level of activity in itself. The theory out forward below is thus not

(Footnote continues on next page)

seen, is assumed to be nonlinear and such that $I'(x)$ is smaller than "normal" both for low and for high levels of economic activity. As explained by Kaldor, this is a plausible assumption given that, on the one hand, at low levels of activity, there is unused capacity so that an increase in activity to a smaller extent leads to an additional construction. On the other hand, there are some factors, such as rising costs of construction, increasing costs and increasing difficulty of borrowing, which cause $I'(x)$ to be small also for unusually high levels of activity. Thus, the investment function will deviate from linearity as shown in Figure 5.8 where (y_1, y_2) is the "range of 'normal' levels of income".

The investment function shown in Figure 5.8., however, is a short-period function in the sense that it assumes a given total amount of fixed equipment. When the latter increases, the range of available investment opportunities restricts and the short-period

(Footnote continued from previous page)

based on this "acceleration principle" (the general validity of which is questionable), but on a much simpler assumption - i.e., that an increase in the current level of profits increases investment demand"

(Kaldor, 1940, p.79f; my emphasis).

For an analogous assumption, involving also in this case an explicit refutation of the accelerator principle, see Kalecki, 1935, 1937.

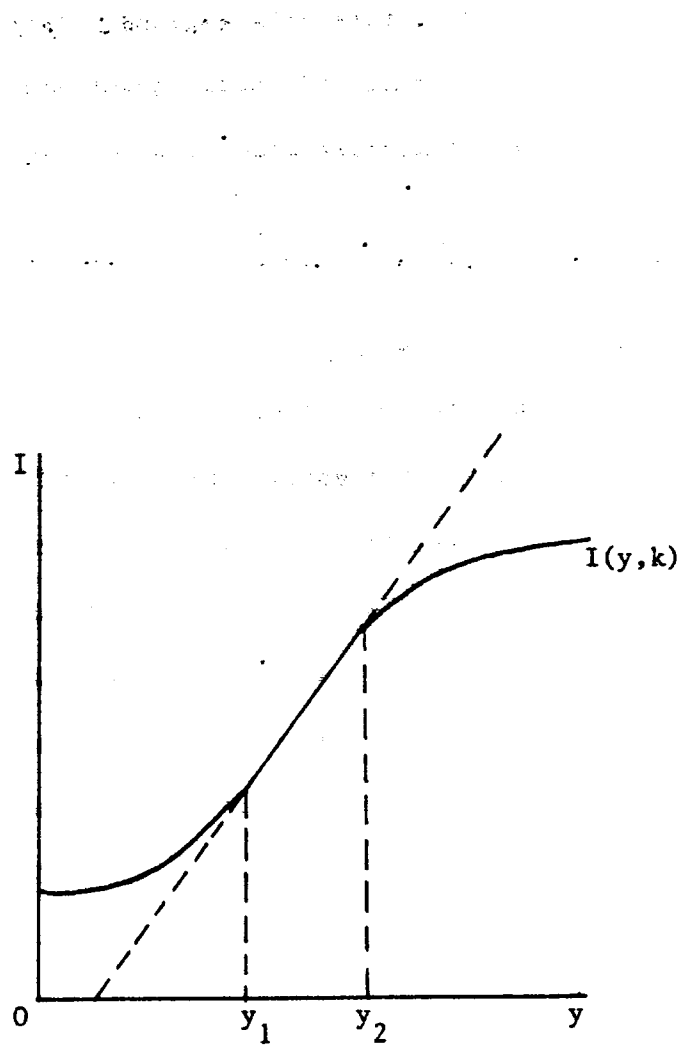


Fig.5.8

function shifts downwards¹. Thus, the essence of this theory is well captured by the standard formulation of Kaldor's investment function we have above considered according to which we have:

$$I = I(y, k),$$

with $I_y > 0$ and $I_k < 0$ ².

In this investment function the crucial nonlinear element is the nonlinear dependence of investment on the level of national income (economic activity). Thus, the negative dependence of investment on the existing capital stock can be represented merely as a shift in the (y, I) plane of the curve representing the nonlinear dependence of investment on income. This is nothing other than Ichimura's representation of Kaldor's investment function we have above considered according to which:

$$I(y, k) = f(y) - mk \quad (5),$$

with $f'(y) > 0$, $f''(y) < 0$, and $m > 0$.

1. In Kaldor's opinion, although there are new inventions which tend to make the investment raise, the negative influence of accumulation is bound to be more powerful. See Kaldor, 1940, p.83.

2. This formulation is slightly different from Kaldor's original one in that it relates investment (and savings) to the level of national income rather than to the level of economic activity measured by employment. The purpose in considering such a formulation is that of putting Kaldor's 1940 model in modern macroeconomic form as stressed, for example, by Chang and Smyth (1971, p.38f). Earlier contributions which consider this formulation are Ichimura, 1954; Bober, 1968, pp.231-241; Evans, 1969; Klein-Preston, 1969.

With this formulation, the function $I(y,k)$ for different levels of the capital stock is as shown in Figure 5.9 where:

$$k_0 < k_1 < k_2,$$

so that, for example for $y = y_0$, we have:

$$I_0(y_0, k_0) > I_1(y_0, k_1) > I_2(y_0, k_2).$$

Given that in Kaldor's paper the conditions for a cycle are obtained by analysing a "simple diagrammatic apparatus", this analytical formulation introduced by Ichimura, and then employed by many other authors, has the useful purpose of allowing a re-examination of Kaldor's theory along more modern lines.

Goodwin's approach to the problem, and as a consequence the reasons which lead him to the formulation of a nonlinear investment function, are on the other hand different. In particular, as we have already noted, his starting point is very different in that the empirical evidence given by some statistical studies according to which the accelerator principle does not correspond to the reality¹ does not induce Goodwin to drop it. Indeed, in Goodwin's opinion, the problem of the questionable "general validity" is not

1. See, for example, Tinbergen, 1938.

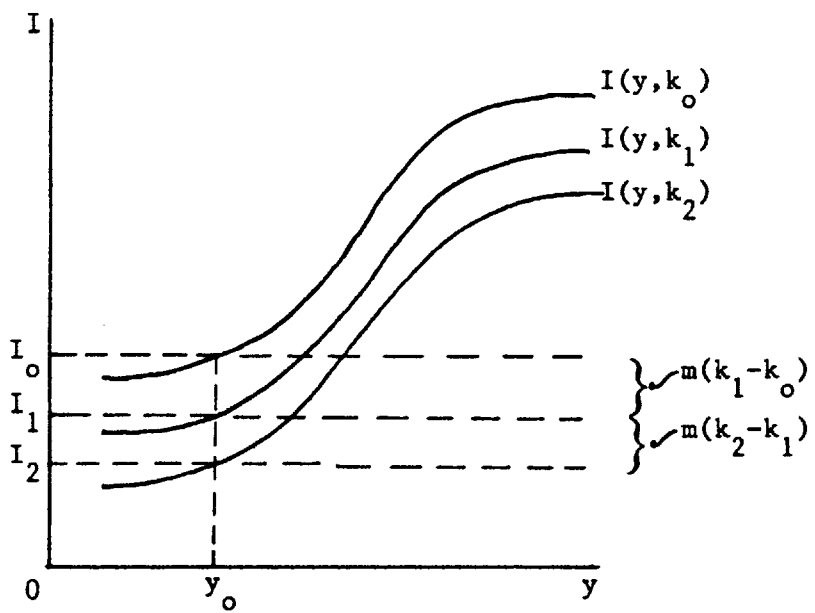


Fig.5.9

with the accelerator principle in itself¹, but with any rigid formulation of it. In reality, there are various reasons why the capital stock is seldom in the desired relation with output. Firstly, the investment goods industry has a given capacity which constitutes an upper limit for the attainable rate of investment; secondly, gross investment cannot be negative so that there is a lower limit to investment corresponding to zero gross investment or, and this amounts to the same, to negative net investment equal to total depreciation. As we have seen, Goodwin takes account of these two limits in choosing a nonlinear, sigmoid shape for the investment function.

It is interesting to note that, as a consequence of these two limits,

"... capital stock cannot be increased fast enough in the upswing, nor decreased fast enough in the downswing"

(Goodwin, 1951, p.230)

Thus, it is possible to give an alternative interpretation of the formulation of Kaldor's nonlinear investment equation given by Ichimura.

1. Indeed, in his opinion, this principle is:

"... merely the statement of a simple consequence of the one omnipresent, incontestable dynamics fact in economics - the necessity to have both stocks and flows of goods"

(Goodwin, R.M., 1951, p.229).

As is well known, according to the "capital stock adjustment principle" proposed by Matthews¹, investment varies directly with the level of national income and inversely with the stock of capital in existence. In continuous terms, we can express this principle in the following way:

$$I = ay - bk \quad (9),$$

where a and b are two positive constants. As we see, (9) is an equation of the type considered in Ichimura's paper where, however, investment depends linearly on national income.

Considering the special case in which, in (9), the ratio a/b is equal to the capital-output ratio v , we obtain what Goodwin, in his 1948 article on the interaction between the multiplier and the acceleration principle, called the "flexible accelerator", namely:

$$\begin{aligned} I &= b(vy - k) = \\ &= b(k^* - k) \end{aligned} \quad (10),$$

where the actual capital stock is assumed to adjust to the desired level (k^*) asymptotically with a speed of adjustment equal to b .

Although according to this interpretation investment also depends linearly on the level of national income, it is possible,

1. In Matthews' 1959 book (pp.40-43) this principle is introduced in the attempt to overcome difficulties implicit in the acceleration principle.

taking account of the above quotation from Goodwin's 1951 article, to give an alternative explanation for the nonlinearity of the investment function. Indeed, given all difficulties which can be encountered in increasing the capital stock fast enough in the upswing or in decreasing it fast enough in the the downswing, we can say that the adjustment of the actual capital stock to the desired level takes place at a speed which varies in the different phases of the cycle. For example, assuming that these phases are distinguished by the level reached by national income, we can write:

$$I = b(y)\{vy - k\}.$$

where $b(y)$, which represents the speed at which the adjustment takes place, is assumed to vary with the level of national income¹.

The impression is that we can find many plausible reasons which may account both in Kaldor's and Goodwin's model for the sigmoid shape of the investment function or, more in general, for the nonlinearity of such a function. In particular, it is important to stress that the sigmoid shape of the investment function is explained in Goodwin's model by limits to the sustainable rate of

1. Although we have once more obtained a nonlinear investment function, however, in this case it is more difficult to account for the sigmoid shape. It would be necessary to introduce additional assumptions - on the first and second derivative of the function $b(y)$ - which would lead us too far from Kaldor's original theory.

investment which find their origins in technological aspects of the process of production of investment goods. Given the simplified framework in which the model is worked out - closed economy model with no government - these limits seem inevitable.

Bearing this in mind, it is interesting to turn our attention to the alternative model proposed by Benassy.

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CHAPTER 6) A "Benassy's Criticism" of the Macro-Behavioural Approach

6.1. Introduction

The models we have analysed in the previous chapter are, in the opinion of both equilibrium and disequilibrium theorists, inevitably condemned to be based on "ad hoc" assumptions. As we have already noted, quoting from Böhm's 1978 and Lucas' 1977 articles, the main reason for this opinion is that these models are based on "macro-behavioural" relations which lack any microeconomic foundation in the traditional sense.

In this regard, it is illuminating now to consider contributions to the problem given by Benassy in which his main purpose is to treat microeconomic concepts and macroeconomic themes in a unified framework in the attempt to construct a bridge between traditional macroeconomics and modern non-Walrasian theory. In particular, it is very relevant for us that one of the traditional topics in

macroeconomics for which this bridge is drawn is the theory of business cycles¹.

To do this, Benassy considers a short-run (nonlinear) model of the IS-LM type in which the combination of destabilizing quantity dynamics and stabilizing price dynamics is shown to generate limit cycles². In Benassy's opinion, his model is, without any doubt, superior with respect to other ("traditional") nonlinear models of the business cycle. The reason for this is that, firstly, in his model the short-run is described rigorously as a non-Walrasian equilibrium, whereas in most existing models there is no such structure; secondly, that to reach conclusions about the cyclical behaviour of the economy, he needs only fairly common assumptions on all functions used, whereas earlier models require rather ad hoc assumptions on these functions, such as sigmoid shapes for the investment function or the Phillips curve.

It is interesting to delve a bit deeper into the solution offered to the problem of the representation of persistent

1. See, for example, Benassy, J.-P., 1984; 1986, Preface, p.xiii and Ch.11, pp.173-185. The same article is published also as a contribution to Day-Eliasson (eds.), 1986, pp.133-145, in which few further comments are added by Benassy.

2. As explained by the author, the stabilizing price dynamics in the model arises from the effect of prices on aggregate demand and wage movements via a traditional Phillips curve, whereas the destabilizing quantity dynamics comes from a traditional investment accelerator and from the dynamic adjustment of demand expectations. In such a model, the dynamics of the economy is represented by a system of two nonlinear differential equations of the first order. Thus, the existence of limit cycles can be proved, as Benassy does, by employing the Poincaré-Bendixson Theorem.

fluctuations, with such an emphasis, by Benassy. To this end, after having presented the model, our intention is to perform an alternative analysis of Benassy's model taking in mind the results we have above obtained while applying the Liénard criterion to models within the macro-behavioural approach.

6.2. Benassy's IS-LM Model: The Short-Run Equilibrium

Benassy's purpose¹ is to achieve a synthesis between models of the non-Walrasian equilibrium type and traditional Keynesians models of the IS-LM type with fixed wage and flexible price².

In such a framework, the IS curve is obtained while taking account of the fact that aggregate demand, which is always equal to sales, is equal to:

$$Y = C(Y,p) + I(X,r) \quad (1),$$

1. See Benassy, 1984; 1986, Ch.11, pp.173-185. Benassy's model is extensively analysed also in Gabisch-Lorenz, 1987, pp.130-138.

2. To understand the grounds on which such a synthesis is based see, Benassy, 1986, Ch.6, "The Three Regimes of the IS-LM Model", pp.103-120. See also Musu, 1986 where the purpose of the author is to show that there exists no conflict between a model of the IS-LM type and the approach of disequilibrium leading to non-Walrasian equilibria. In these contributions, both authors give importance to the case in which there is equilibrium in the goods market and disequilibrium, with unemployment, in the labor market.

where Y is income (output); p , the price; X , the expected demand; r , the interest rate; and where all usual assumptions on the signs of the derivatives of the various functions are made, namely, $C_y > 0$, $C_p < 0$, $I_x > 0$, and $I_r < 0$.

Firms, which are assumed to be not rationed at all, have a demand for labor equal to their notional demand and, as a consequence, also a short-run supply of goods equal to their notional supply. Thus, given the short-run production function with decreasing returns which, for a given level of the capital stock, relates output to the level of employment,

$$Y = F(L),$$

with $F' = dF/dL > 0$ and $F'' = d^2F/dL^2 < 0$, we have that:

$$Y = F(F'^{-1}(w/p)) = S(w,p) \quad (2),$$

where w is the money wage and where $S_w < 0$ and $S_p > 0$ ¹.

1. These partial derivatives of the function $S(w,p)$ are easily obtained by considering the maximization problem solved by the firm. Writing Π for profits, this problem is the following:

$$\text{Max } \Pi = pF(L) - wL.$$

The first order condition for a maximum is:

$$\frac{w}{p} = F'(L).$$

(Footnote continues on next page)

Thus, the IS curve is given by:

(Footnote continued from previous page)

Given that $F'' < 0$ for all L , we can use the inverse function rule and write:

$$L = F'^{-1}\left(\frac{w}{p}\right) = H(w, p).$$

From the first order condition, we find:

$$\frac{\partial(w/p)}{\partial w} = \frac{1}{p} = \frac{dF'(L)}{dL} \frac{\partial L}{\partial w} = F''(L) \frac{\partial H(w, p)}{\partial w} = F''(L) H_w,$$

and

$$\frac{\partial(w/p)}{\partial p} = \frac{-w}{p^2} = \frac{dF'(L)}{dL} \frac{\partial L}{\partial p} = F''(L) \frac{\partial H(w, p)}{\partial p} = F''(L) H_p,$$

from which:

$$H_w = \frac{1}{pF''(L)} < 0.$$

and:

$$H_p = -\frac{w}{p^2 F''(L)} > 0.$$

Thus, the short-run equilibrium production function is:

$$Y = F(F'^{-1}(w/p)) = F(H(w, p)) = S(w, p),$$

such that:

$$\frac{\partial Y}{\partial w} = \frac{dF}{dL} \frac{\partial H}{\partial w} = S_w < 0,$$

and:

$$\frac{\partial Y}{\partial p} = \frac{dF}{dL} \frac{\partial H}{\partial p} = S_p > 0.$$

$$Y = S(w, p) = C(Y, p) + I(X, r)^1 \quad (3).$$

With regard to the LM curve, it is assumed that it is of the form:

$$L(Y, r, p) = \bar{M} \quad (4),$$

where $L_Y, L_p > 0, L_r < 0$ and where \bar{M} is the (fixed) quantity of money in the economy².

Thus, the short-run equilibrium values of the variables are determined by the following system:

1. The slope of the IS curve can be calculated taking the total differential of both sides of (1) and (2):

$$dY = C_Y dY + C_p dp + I_X dX + I_r dr \quad (1'),$$

$$dY = S_w^Y dw + S_p^P dp \quad (2').$$

In the short run, w and X are given so that, from (2'), we obtain:

$$dp = \frac{1}{S_p} dY,$$

from which, inserting in (1'):

$$\left(\frac{dr}{dY}\right)_{IS} = \frac{1 - C_Y - (C_p/S_p)}{I_r},$$

which, if we assume that $1 > C_Y + (C_p/S_p)$, is negative.

2. The slope of the LM curve in the (y, r) plane is:

$$\left(\frac{dr}{dY}\right)_{LM} = \frac{-(L_Y + L_p/S_p)}{L_r} > 0.$$

$$Y = C(Y,p) + I(X,r) \quad (1),$$

$$Y = S(w,p) \quad (2),$$

$$L(Y,r,p) = \bar{M} \quad (4).$$

In the case considered by Benassy, with fixed wage and flexible price, the above system determines the equilibrium values of Y , r , and p for given x and w . Thus, its solution in Y is a relation of the type:

$$Y = z(X,w) \quad (5).$$

the two partial derivatives of which can be easily calculated by using the implicit function rule.

In (1), (2), and (4), we have:

$$Y - C(Y,p) - I(X,r) = F_1(Y,r,p;X,w) = 0,$$

$$Y - S(w,p) = F_3(Y,r,p;X,w) = 0,$$

$$L(Y,r,p) - \bar{M} = F_2(Y,r,p;X,w) = 0.$$

Thus, taking the total differential of both sides for each of the three equations, we obtain relations of the type:

$$\begin{aligned} \left(\frac{\partial F_i}{\partial Y}\right)dY + \left(\frac{\partial F_i}{\partial r}\right)dr + \left(\frac{\partial F_i}{\partial p}\right)dp = \\ = -\left[\left(\frac{\partial F_i}{\partial X}\right)dX + \left(\frac{\partial F_i}{\partial w}\right)dw\right], \quad (i=1,2,3). \end{aligned}$$

Calculating all partial derivatives, we obtain the following system:

$$(1 - C_y)dY - I_r dr - C_p dp = I_x dx,$$

$$dY - S_p dp = S_w dw,$$

$$L_y dY + L_r dr + L_p dp = 0,$$

the Jacobian of which is:

$$|J| = \begin{vmatrix} (1 - C_y) & -I_r & -C_p \\ 1 & 0 & -S_p \\ L_y & L_r & L_p \end{vmatrix} =$$

$$= I_r L_p - C_p L_r + S_p (1 - C_y) L_r + S_p I_r L_y < 0.$$

Supposing, for example, that $dw = 0$, the system becomes:

$$\begin{bmatrix} (1 - C_y) & -I_r & -C_p \\ 1 & 0 & -S_p \\ L_y & L_r & L_p \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial x} \\ \frac{\partial r}{\partial x} \\ \frac{\partial p}{\partial x} \end{bmatrix} = \begin{bmatrix} I_x \\ 0 \\ 0 \end{bmatrix},$$

from which, using the Cramer's rule, we obtain:

$$\frac{\partial Y}{\partial x} = \frac{\partial z(X, w)}{\partial x} = z_x = \frac{\begin{vmatrix} I_x & -I_r & -C_p \\ 0 & 0 & -S_p \\ 0 & L_r & L_p \end{vmatrix}}{|J|} =$$

$$= \frac{I_x L_r S_p}{|J|} > 0.$$

Following an analogous procedure, assuming that $dw \neq 0$ and $dX = 0$, we find:

$$\begin{aligned} \frac{\partial Y}{\partial w} &= \frac{\partial z(X, w)}{\partial w} = z_w = \frac{\begin{vmatrix} 0 & -I_r & -C_p \\ S_w & 0 & -S_p \\ 0 & L_r & L_p \end{vmatrix}}{\begin{vmatrix} S_w & (-I_r L_p + C_p L_r) \end{vmatrix}} / |J| = \\ &= -\frac{S_w (-I_r L_p + C_p L_r)}{J} < 0. \end{aligned}$$

To sum up, the result of the analysis carried out by Benassy in the first part of his paper are equilibrium solutions of the short run IS - LM system parameterized by w and X . The crucial step is then seen to be that of specifying the two equations governing the behaviour in time of these two variables. Our purpose now is to analyse the consequences which the formulation chosen by Benassy has for the dynamics of the economy in the long period.

6.3. Equations Governing the Dynamics of the System

With regard to the nominal wage, Benassy assumes that it evolves according to a Phillips Curve of the type:

$$\dot{w} = H(u), \quad H' < 0 \tag{6}.$$

where, writing L_0 for the (inelastic) supply of labour, the level u of unemployment is given by:

$$u = L_0 - F^{-1}(Y).$$

As is shown in Figure 6.1, the function H is assumed to be such that¹:

$$\lim_{u \rightarrow 0} H(u) = +\infty,$$

$$H(\bar{u}) = 0.$$

i.e., the formulation chosen by Benassy takes account of the assumption according to which the firm is not constrained on the labour market implying that full employment is never reached.

From (6), it is possible to obtain a relation between the increase in nominal wage and output. Indeed, we have:

$$\begin{aligned} \dot{w} &= H(u) = H(L_0 - F^{-1}(Y)) = \\ &= G(Y), \quad G'(Y) > 0 \end{aligned} \quad (7),$$

where, writing Y_0 for the full employment output $F(L_0)$ and \bar{Y} for $F(L_0 - \bar{u})$, the function G , as is shown in Figure 6.2, is such that:

1. See Benassy, 1984, pp.78-79.

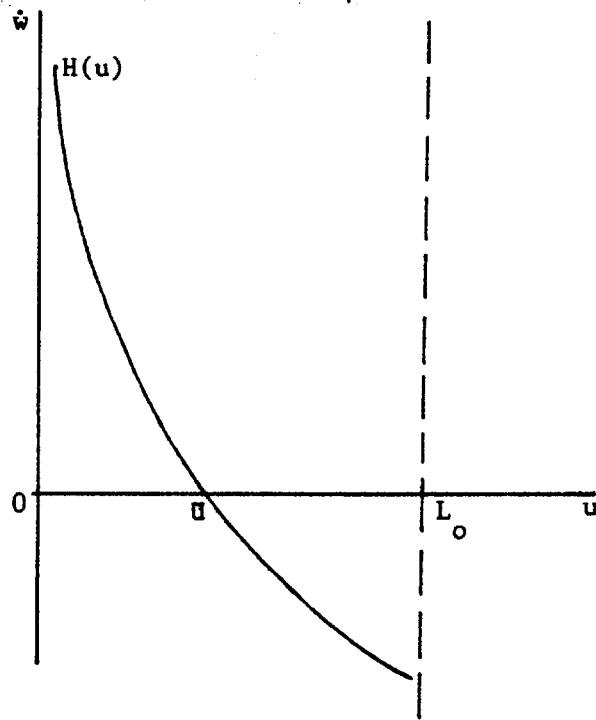


Fig.6.1

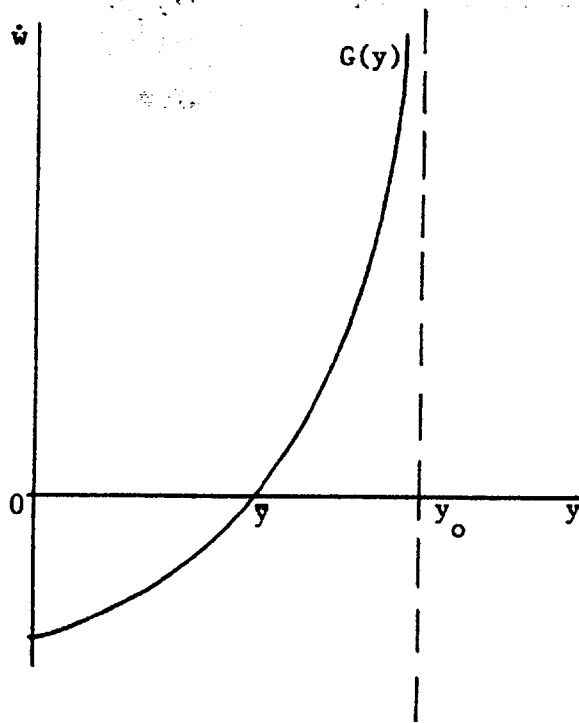


Fig.6.2

$$\lim_{Y \rightarrow Y_0} G(Y) = \lim_{Y \rightarrow Y_0} H(L_0 - F^{-1}(Y)) = +\infty,$$

$$G(\bar{Y}) = H(L_0 - F^{-1}(Y))0 = H(\bar{u}) = 0.$$

Finally, with regard to the dynamics of the expected demand X , Benassy assumes that it "adapts" to actual demand with a speed of adjustment equal to μ , i.e.:

$$\dot{X} = \mu(Y - X), \quad \mu > 0 \quad (8).$$

Thus, the system describing the dynamics of the model economy is the following:

$$\dot{w} = G(Y),$$

$$\dot{X} = \mu(Y - X),$$

where:

$$Y = z(X, w).$$

Writing (Y^*, X^*, w^*) for the long-run equilibrium values of the variables, we find that:

$$Y^* = X^* = \bar{Y}.$$

Moreover, w^* is the value of w which satisfies the relation:

$$z(\bar{Y}, w^*) = \bar{Y}.$$

Benassy's contention is that this long-period equilibrium, which exists and is unique¹, is either stable or unstable, in which case there will be at least one limit cycle.

As shown by him², the equilibrium point is locally unstable or

1. See Benassy, J.-P., 1986, p.180.

2. In order to study the local stability of the equilibrium, it is necessary to consider the following linearized system:

$$\begin{aligned}\dot{w} &= [G'(\bar{Y})z_w][w - w^*] + [G'(\bar{Y})z_x][X - X^*], \\ \dot{X} &= (\mu z_w)[w - w^*] + [\mu(z_x - 1)][X - X^*],\end{aligned}$$

the characteristic roots of which are:

$$\lambda_{1,2} = (1/2)\{\text{Tr}(A) \pm [\text{Tr}^2(A) - 4\text{Det}(A)]^{1/2}\},$$

where:

$$A = \begin{bmatrix} G'(\bar{Y})z_w & G'(\bar{Y})z_x \\ \mu z_w & \mu(z_x - 1) \end{bmatrix},$$

so that:

$$\text{Tr}(A) = G'(\bar{Y})z_w + \mu(z_x - 1) \geq 0,$$

$$\text{Det}(A) = G'(\bar{Y})z_w \mu(z_x - 1) - G'(\bar{Y})z_x \mu z_w = -G'(\bar{Y})z_w > 0.$$

Thus, given that $\text{Det}(A)$ is always positive, the stability of the equilibrium point is determined by the sign of $\text{Tr}(A)$: if $\text{Tr}(A) < 0$ (> 0), the equilibrium is locally stable (unstable). See Benassy, 1984, p.84.

locally stable according as to whether:

$$\mu(z_x - 1) > -G'(\bar{Y})z_w \quad (9),$$

or

$$\mu(z_x - 1) < -G'(\bar{Y})z_w \quad (10).$$

On the basis of the assumptions made by Benassy, we can conclude that the partial derivative of Y with respect to expected demand is positive. However, it is not possible to say something about its absolute value, namely, whether it is less or greater than one. Thus, there are different, alternative cases which must be considered. For example, when z_x is less than one, the left-hand side of the condition is negative so that (10) holds and the equilibrium point is stable. However, when z_x is greater than one, there are two possibilities depending on whether the positive difference $(z_x - 1)$ is less or greater than the right-hand side of the conditions we have obtained; i.e.:

$$z_x - 1 > 0,$$

is a necessary - although not sufficient - condition for the equilibrium point to be unstable.

The analysis of the stability of the equilibrium point has an important relation with the analysis of the $\dot{X} = 0$ and $\dot{w} = 0$ loci in the phase diagram.

With regard to the locus $\dot{w} = 0$, we have:

$$(1) \dot{w} = 0 = G(Y) \longleftrightarrow Y = z(X, w) = \bar{Y} \longleftrightarrow F_1(X, w) = 0.$$

Implicitly, this defines a relation between X and w of the type:

$$X = f_1(w).$$

We have:

$$dF_1 = 0 = z_x dX + z_w dw,$$

from which:

$$\frac{dX}{dw} / \dot{w}=0 = \frac{df_1(w)}{dw} = f'_1(w) = -\frac{z_w}{z_x} > 0;$$

whereas, with regard to the locus on which $\dot{x} = 0$:

$$(2) \dot{x} = 0 = \mu(Y - X) \longleftrightarrow z(X, w) = X \longleftrightarrow F_2(X, w) = 0.$$

Implicitly, this defines a relation of the type:

$$X = f_2(w).$$

We obtain:

$$dF_2 = 0 = (z_x - 1)dX + z_w dw,$$

from which:

$$\frac{dX}{dw} / \dot{x}=0 = \frac{df_2(w)}{dw} = f'_2(w) = \frac{-z_w}{(z_x - 1)}.$$

Thus:

$$\frac{dX}{dw} / \dot{x}=0 \geq 0,$$

according as to whether:

$$z_x - 1 \geq 0.$$

Thus, we can express the condition for the instability of the equilibrium point in terms of the slope of the locus $\dot{X} = 0$ and say that a necessary, although not sufficient, condition for the equilibrium point to be unstable is that the locus $\dot{X} = 0$ is upward sloping at (w^*, X^*) , for example, as shown in Figure 6.3. If this is the case, Benassy shows, using the Poincaré-Bendixson Theorem, that there exists at least a limit cycle¹.

Thus, in Benassy's model, the crucial role in the generation of limit cycles is played by the specification of a nonlinear dependence of the equilibrium level of income on expected demand.

To understand this, it is useful to apply to Benassy's model the same Liénard's criterion we have earlier applied to Goodwin's and Kaldor's models.

6.4. Sufficient Condition for a Limit Cycle in a Simplified Version of Benassy's Model

In order to concentrate our attention on the crucial nonlinearity of Benassy's model - i.e., that according to which $z_x \geq 1$ at different levels of expected demand - we consider a simplified version of the model.

1. See, for example, Benassy, 1984, pp.86-88.

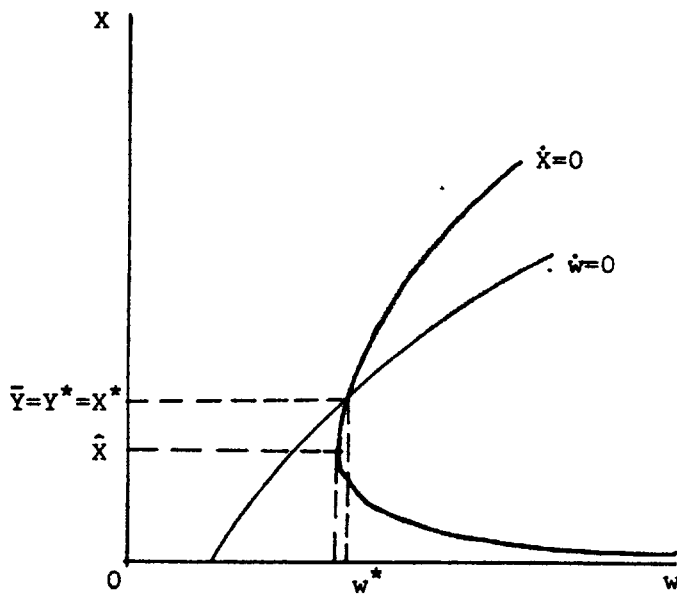


Fig.6.3

To this end, we replace the Phillips curve we have considered in the previous section with its linear approximation which is obtained expanding the function $G(Y)$ in Taylor's series at the equilibrium point and then neglecting all terms of order higher than the first.

In doing this, the dynamical system determining the evolution of the economy becomes:

$$\dot{w} = G'(\bar{Y})(Y - \bar{Y}) \quad (6'),$$

$$\dot{X} = \mu(Y - X) \quad (7),$$

where:

$$Y = z(X, w) \quad (8).$$

To apply Liénard's criterion for the existence of a limit cycle, we must reduce the dynamical system given by the two first order differential equations (6') - (7), where Y is given by (8), to an equivalent differential equation of the second order.

To do this, we take the time derivative of both sides of (8) so as to obtain:

$$\begin{aligned} \dot{Y} &= z_x \dot{X} + z_w \dot{w} = \\ &= \mu z_x (Y - X) + z_w G'(\bar{Y})(Y - \bar{Y}) = \\ &= \mu z_x (y - x) + z_w G'(\bar{Y})y \end{aligned} \quad (9),$$

where:

$$y = Y - \bar{Y} \text{ and } x = X - X^*.$$

Thus, inserting in (9) the expression for y which is given by (7), we obtain the following nonlinear differential equation of the second order in x :

$$\ddot{x} + \{\mu(1 - z_x) - z_w G'(\bar{Y})\}\dot{x} - \mu z_w G'(\bar{Y})x = 0 \quad (10).$$

To simplify still further, we assume that all functions (1), (2) and (3) are linear in all arguments except the investment function which is nonlinear in the expected demand with:

$$\frac{\partial^2 I}{\partial x^2} = I_{xx} \neq 0,$$

and:

$$\frac{\partial^2 I}{\partial x \partial r} = \frac{\partial^2 I}{\partial r^2} = 0.$$

From this assumption, it follows that the short-run equilibrium value of $Y = z(X, w)$ is such that its partial derivative with respect to X depends only on X whereas its partial derivative with respect to w is constant. As a consequence, we can write:

$$z_x = Z'(X) > 0,$$

$$z_w = \bar{z}_w < 0 \text{ (constant),}$$

and:

$$z_{xx} = z''(X) \neq 0,$$

$$z_{ww} = z_{xw} = 0^1.$$

Thus, equation (10) becomes:

$$\ddot{x} + \{\mu[1 - z'(X)] - \bar{z}_w G'(\bar{Y})\}\dot{x} - \mu \bar{z}_w G'(\bar{Y})x = 0 \quad (10'),$$

i.e.:

$$\ddot{x} + f_2(x)\dot{x} + g_2(x) = 0 \quad (11),$$

1. In the case in which all functions (1), (2), and (3) are linear in all arguments except the investment function which is nonlinear in the expected demand variable, we obtain:

i.e.:

$$J \cdot \begin{bmatrix} d^2 y \\ d^2 r \\ d^2 p \end{bmatrix} = \begin{bmatrix} I_{xx} dx^2 \\ 0 \\ 0 \end{bmatrix},$$

where J is the same matrix of coefficients we have considered above. Thus, $\partial^2 y / \partial x^2$ is different from zero and such that:

$$\partial^2 y / \partial x^2 = z_{xx} = -I_{xx} L_r S_p / |J|$$

For an analogous simplifying assumption, see Schinasi, G.J., 1981.

where:

$$f_2(x) = \mu[1 - Z'(x+X^*)] - \bar{z}_w G'(\bar{Y}),$$

$$g_2(x) = -\mu \bar{z}_w G'(\bar{Y})x,$$

which is an equation of (generalized) van der Pol's type.

By using the Poincaré-Bendixson Theorem, Benassy shows that the dynamical system (6')-(7) admits as a solution a limit cycle representing persistent fluctuations in the economy. However, in spite of Benassy's claim, we can conclude while taking account of the analysis we have developed in the previous chapter with regard to models within the macro-behavioural approach, that this result is based on an "ad hoc" assumption in the same sense in which those made in Kaldor's and Goodwin's models are.

To understand this, it is useful to consider the same Liénard criterion we have above employed to obtain sufficient conditions for the existence of limit cycles in Goodwin's and Kaldor's models. To do this, let us write equation (10') as:

$$\dot{x} = u \tag{10'a},$$

$$\dot{u} = -f_2(x)\dot{x} - g_2(x) \tag{10'b}.$$

Thus, in Benassy's dynamical system, the total energy is given by:

$$E = \frac{\mu}{2} \dot{x}^2 + G_2(x),$$

where:

$$G_2(x) = \int_0^x g_2(x)dx,$$

so that:

$$\frac{dE}{dt} = -f_2(x)u^2 \begin{matrix} > \\ < \end{matrix} 0,$$

according as to whether:

$$f_2(x) = u[1 - z'(x)] - \bar{z}_w G'(\bar{Y}) \begin{matrix} < \\ > \end{matrix} 0.$$

Having seen this, it is possible to understand the cruciality of the assumption implicitly made by Benassy according to which there exists a value of X - say \hat{X} - for which $z_x = 1^1$. Given this assumption, we know that there exists an $X = \hat{X} < X^*$ for which we have:

$$f(\hat{X}) = -\bar{z}_w G'(\bar{Y}) > 0.$$

On the other hand, we have seen above that the condition for the equilibrium point to be unstable requires that:

1. That such an assumption is made by Benassy is implicit in the phase diagram where he draws, for the unstable case, the graph of the locus $\hat{X} = 0$. See Benassy, 1986, pp.182-183.

$$f(X^*) = \mu[1 - Z'(X^*)] - \bar{z}_w G'(\bar{Y}) < 0.$$

Moreover, given the proof of the existence of a limit cycle offered by Benassy by using the Poincaré-Bendixson theorem, we know that the forms chosen for the various functions involved in the model are such that these positive and negative exchanges of energy compensate each other exactly at least once over a trajectory.

Thus, although Benassy is surely correct in saying that in his model he does not need to assume a sigmoid shape for any of the function involved, his solution is subject to the same type of criticism.

Some conclusions can be drawn from the analysis we have developed.

As we have seen in the previous chapter, both Kaldor and Goodwin in their papers are mainly concerned with the specification of the investment function. This should hardly be surprising given the crucial role that factors explaining the investment behaviour have in determining the dynamics of an economy. In particular, as we have already stressed, both authors give explanations for the reasons behind the sigmoid shape of the investment function which are plausible if seen as part of the very simplified framework ("closed economy model with no role for government") in which they are made.

On the contrary, assumptions are not often explicitly made by Benassy. In particular, he considers it an advantage of his model that cyclical fluctuations in economic activity are obtained without having to assume any "ad hoc" shape for the various

functions involved. However, taking account of the analysis we have above developed, it is difficult to understand what the reasons are for the superiority of such an approach. For example, we have seen that the crucial assumption for the existence of limit cycles is that according to which the partial derivative of Y with respect to X is positive but less than one for "low" levels of expected demand and greater than one for "high" levels of expected demand. Although no word is spent by Benassy on this matter, this implies that, to have a limit cycle, we must have $\partial^2 Y / \partial X^2 = z_{xx} > 0$. In the simplified model we have considered as well as in the original model, this is the case when $I_{xx} > 0$. However, to require that I_{xx} is positive for all levels of expected demand is a fairly "ad hoc" assumption which would seem to deserve, at least, some discussion¹.

The lack of attention given to the analysis of the investment process makes itself felt also in another important respect. As we have seen, Benassy's model has the "traditional" structure of short run non-Walrasian equilibrium which is not abandoned even when long run considerations are introduced into the model. For example, given the assumed short run structure, the production function relates output to the level of employment for a given level of the capital stock. The resulting short run equilibrium level of output (income) is then used, together with the two equations describing the behaviour in time of w and x , to obtain the dynamical system

1. An analogous comment is made in Gabisch-Lorenz, 1987, p.138.

which governs the dynamics of the economy. The problem is that, although this is surely a "traditional" assumption for the short run, it is difficult to understand what it means to maintain it in the long run. As a result, we have an investment function which has an important role in determining some features of the dynamics of the economy but which has no effect on the level of the stock of capital and, therefore, not even on the level of output. This "shortcoming" of his model is noted by Benassy himself, who, to give a rationale to it, appeals to the limits of "available mathematical cycle theory"¹.

Again, the impression is that very little attention is paid to questions posed by authors advocating other approaches and to the developments in business cycle modelling to which these questions have lead and are leading. If nothing else, this would seem to be a necessary first step if the purpose is really to improve our "understanding of business cycles".

1. For example, in his contribution to Day-Eliasson's volume, he say that in his model, he

"... could have added an equation depicting the evolution of the capital stock, so that investment would have effects on both the demand and supply sides ... The inclusion of these variables, and possibly other relevant ones, would have however increased the dimensionality of our dynamic system beyond that which can be elegantly handled by available mathematical cycle theory"

(ib., p.144; my emphasis).

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CHAPTER 7) Alternative Approaches to the Explanation of Macro- Economic Fluctuations: Some Conclusions

In the opinion of the authors who have developed the macro-behavioural approach to economic dynamics, the fluctuations which have characterized and continue to characterize the dynamics of the capitalist system are endogenous and explained by the same factors which account for the growth of the system. Starting from this point of view, there are two problems which must be solved in constructing a model. Firstly, it is necessary that the model describe the mutual conditioning of growth and cycle. Secondly, having achieved this result, it is necessary that the resulting fluctuations are persistent.

As is well known, both problems cannot find a satisfactory solution in a linear model. This is shown, for example, by the results which have been obtained in linear models of the multiplier-accelerator type where the trend is simply superimposed on the cyclical part of the model and where the case of fluctuations of constant amplitude, which neither die out nor explode, is structurally unstable. Historically, the awareness of

this shortcoming of linear models has led to the consideration of nonlinearities in the relations among the crucial variables and the advantages which this has brought about are obvious to everybody.

Thus, in this sense, we can say that the order of priority has been the proper one in that the mathematical tool used for the formalisation of the model has been adapted to the economic problem under study and not the other way around. Given that the purpose was to represent the pervasive and endogenous cyclical fluctuations which characterize the growth of capitalist economies, it has been understood that it was necessary to use nonlinear mathematical tools which are the only ones which allow one to attack the problem.

The problem, however, as we have seen while analysing Goodwin's model, is that the priority which is given to the economic content is obscured by the various simplifications and approximations which are introduced into the model in order to ensure that the final equation describing the dynamics of the economy is a specific nonlinear differential equation (e.g. van der Pol's equation). From this point of view, these simplifications and approximations are certainly "ad hoc" in the sense that they are made to reach a pre-determined analytical form for the final equation. Clearly, they are made with the purpose of having the possibility of exploiting all advantages offered by the mathematical tool rather than with the purpose of improving our understanding of business cycles.

From the analysis we have developed in Part 1 and in Chapter 6, it is evident that the cycle-trend problem and the problem of the

representation of persistent fluctuations are both neglected by authors advocating the most recent approaches and that this has important consequences with regard to the developments and new insights which we can expect from these approaches. As we have seen, on the one side, in the standard formulation of the equilibrium approach, the persistence of economic fluctuations is explained without any role for endogenous factors. In Lucas' log linear model, for example, the trend component of output is simply superimposed on the cyclical component which is explained only in terms of exogenous shocks. In Grandmont's model, on the other hand, the cycle is endogenous. However, one has the impression that more attention is paid to "mathematical elegance" than to economic content in that the latter is manipulated to fit the mathematical tool. As we have seen, Grandmont's purpose is to show that an equilibrium model can describe irregular fluctuations even in the deterministic case. As is well known, deterministic irregular fluctuations can be described as a solution to nonlinear difference equations. For this reason, Grandmont's problem becomes that of specifying the model in such a way that all results already obtained in chaotic dynamics theory for nonlinear difference equation of the first order can be exploited. When it is assumed that economic agents live only two periods, the very abstract overlapping generations framework ensures that the final equation is a difference equation of the first order. Then, assuming an "ad hoc" nonlinearity for one of the functions involved, the required mathematical result is easily achieved. However, notwithstanding the highly sophisticated and elegant techniques which the analysis

of such a model requires, the consequence of all this is that the model is too simple to have any possibility of giving us new insights into the problem of the explanation of persistent fluctuations. As we have seen, moreover, the choice of concentrating attention on the "backward" equation is explained by a pure mathematical reason rather than by economic arguments.

On the other side, in the disequilibrium approach attention is not concentrated on the factors explaining the investment behaviour and this seems to reduce the importance of contributions which otherwise could be thought to be of interest.

In Benassy's model, for example, all factors which determine the behaviour in time of the capital stock are deliberately ignored and the analysis of them is postponed until future research. In doing this, the purpose is to avoid that the dimensionality of the system is greater than two so that, once again, the impression one has is that more importance is given to "mathematical elegance" - which for a system of the second order is ensured by the use of Poincaré-Bendixson's theorem - than to economic content. As a result, in the attempt to overcome "ad hoc" assumptions made in other models of the business cycle, Benassy offers an alternative model to which the criticism of "ad hoc" assumptions applies in the same sense however. Worse still, the consequence of this is that he concentrates his attention on a problem which, given the simplified framework in which Goodwin's and Kaldor's models are developed, is not at the centre of the matter.

In order to compare the alternative approaches to business cycle modelling, we have concentrated our attention on the reciprocal accusations of the "ad hoc" nature of the assumptions made in the various approaches. This is what has allowed us to describe the most recent contributions to the equilibrium approach as attempts to overcome the "ad hoc" assumptions made in the Lucasian model and to see Benassy's model as an attempt to overcome those made in other models, for example, in Goodwin's and Kaldor's models. In particular, to evaluate the relevance of Benassy's criticism of the macro-behavioural approach, we have limited our attention to the problem of understanding the sense in which the choice of a sigmoid shape for the investment function in Goodwin's and Kaldor's model is "ad hoc". However, it seems that it would be more important to concentrate on another more basic simplification which is performed by Goodwin and which is usually not mentioned - perhaps because it has to do with the dimensionality of the system.

As we have already stressed, a crucial problem which must be taken into account in constructing a dynamic model of the economy is that concerning the interaction between cycle and growth. Concentration on this problem in recent years has led authors, taking as a starting point Goodwin's 1967 model, to consider a different ("growth cycle") framework in which the two crucial variables are the rate of unemployment and the (functional) distribution of income. To have the possibility of comparing the alternative approaches to business cycle theory by considering the reciprocal accusations of the "ad hoc" nature of the various

assumptions, we have deliberately ignored this important part of the literature. However, it seems possible to learn something from the way in which authors have tried to develop and continue to try to develop and improve the original model. Goodwin's 1967 model is formulated in terms of Lotka-Volterra's equations. As was the case for Goodwin's 1951 model, the purpose of obtaining a representation in terms of such equations requires the introduction of many simplifications such as the consideration of a linear Phillips curve or the assumption of a constant capital-output ratio. More crucially, it is necessary to assume that the goods market is in equilibrium. These are all "ad hoc" assumptions in the sense that they are made to ensure that the analysis of the resulting dynamics can be made using Volterra's graphical method. To overcome these "ad hoc" assumptions, the model has been recently reformulated in various directions. Among other things, it has been shown that, with disequilibrium also in the goods market, the resulting dynamical system is of the third order and that, even without assuming any specific functional form for the various functions involved, it is possible to obtain important qualitative results by using the Hopf bifurcation theorem. In other words, we already have examples of what, in Benassy's opinion, should be postponed until future research.

Bearing this in mind, it is possible to understand the crucial nature of another assumption which is made by Goodwin in his 1951 model. As we have seen, although the necessity of a nonlinear formulation is strongly advocated by Goodwin in particular because of the advantages which this kind of formulation has for the

representation of the interaction between cycle and trend, it is not possible to analyse this interaction in his nonlinear model. The reason for this is that Goodwin assumes that all autonomous outlays are constant. Although he introduces this assumption only saying that

"... For the moment we may take $O^*(t)$ to be a constant,
 O^* "

(Goodwin, 1951, p.12)

it is easy to understand that this is the crucial simplification which allows Goodwin to write the equation in terms of deviations from equilibrium and then to perform the subsequent analysis using the well known Poincaré-Liénard method of graphical integration.

With the equation written in his original form, we have:

$$\epsilon \ddot{Y} + (\epsilon + s)\dot{Y} - \phi(\dot{Y}) + sY = l(t+1),$$

i.e., time enters explicitly in the right-hand side of the equation and this causes the dimensionality of the system to increase.

Innovational investment is part of $l(t+1)$, i.e., of that component of investment outlays which in the model is taken as exogenous. Assuming, for example, that innovational investment is a periodic function of time of the following type:

$$l(t+1) = \cos \omega(t+1),$$

the nonlinear equation is equivalent to the following third order system:

$$\dot{Y} = x,$$

$$\dot{x} = -\frac{1}{\varepsilon}[\psi(x) + sY] + \frac{1}{\varepsilon} \cos \theta,$$

$$\dot{\theta} = \omega.$$

As is well known, in this three dimensional system, we may have an infinite set of new phenomena in addition to the equilibrium point and the (stable) limit cycle which can appear from the planar theory of nonlinear oscillations.

Thus, the crucial "ad hoc" assumption (which is made in order to allow an analysis of the model by exploiting all results available for two dimensional oscillatory systems) is the one according to which the autonomous component of investment outlays is constant, i.e., independent of time. If this was acceptable in the late forties/early fifties when, in spite of Le Corbeiller's and Hamburger's advice, the use of nonlinear tools in economic modelling was still in its infancy, it is much less so nowadays.

As a consequence, it is puzzling that in very recent contributions to business cycle modelling, in making the truly "ad hoc" assumption according to which investment has no effect on the level of the stock of capital, appeal is made to the necessity of keeping the dimensionality of the system beyond that which can be "elegantly handled by available mathematical cycle theories".

The only explanation which can be given for such a situation is that the approach we have referred to as the macro-behavioural approach and the disequilibrium (or equilibrium with rationing) approach have been developed with no reciprocal consideration at all. This is perhaps inevitable given that, as we have stressed in the introductory chapter, the points of departure of the two approaches are very different in that one is macro-dynamic from the very outset whereas the other attempts to become so starting from a static choice-theoretic framework.

As a consequence, notwithstanding their common stress on "disequilibrium", they were, and must continue to be seen, as alternative approaches.

[illegible]

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